MATHEMATICS

2019 - 2020

For Preparatory Year Two
First Term
Student’s Book
Authors

Mr. Omar Fouad Gaballa

Prof. Dr. Afaf Abo-ElFoutoh Salah  Dr. Essam Wasfi Roupheael

Mr. Mahmoud Yasser Al-Khatieb  Mr. Serafiem Elias Iskander

Revised by Mr. Samir Mohamed Sedawy
Dear students:

It is extremely great pleasure to introduce the mathematics book for second preparatory. We have been specially cautious to make your learning to the mathematics enjoyable and useful since it has many practical applications in real life as well as in the other subjects. This gives you a chance to be aware of the importance of learning mathematics, to determine its value and to appreciate the mathematicians roles.

This book sheds new lights on the activities as a basic objective. Additionally, we have tried to introduce the subject simply and excitingly to help attaining mathematical knowledge as well as gaining the patterns of positive thinking which pave your way to creativity.

This book has been divided into units, each unit contains lessons. Colors and pictures are effectively used to illustrate some mathematical concepts and the properties of figures. Lingual level of previous study has been taken into consideration.

Our great interest here is to help you get the information by your self in order to develop your self-study skills.

Calculators and computer sets are used when there's a need for. Exercises, practices, general exams, portfolios, unit test, general tests, and final term tests attached with model answers have been involved to help you review the curriculum completely.

Eventually, we hope getting the right track for the benefits of our students as well as for our dearest Egypt hoping bright future to our dearest students.

Authors
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<tr>
<td>Z</td>
<td>The set of integer numbers</td>
</tr>
<tr>
<td>Q</td>
<td>The set of rational numbers</td>
</tr>
<tr>
<td>Q′</td>
<td>The set of irrational numbers</td>
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<tr>
<td>R</td>
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<td>Cube root of number a</td>
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<td>Closed interval</td>
</tr>
<tr>
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<td>Open interval</td>
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<tr>
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<td>Half-open (closed) interval</td>
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<tr>
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<td>⊥</td>
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<td>parallel to</td>
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<td>AB</td>
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<td>AB</td>
<td>Ray AB</td>
</tr>
<tr>
<td>AB</td>
<td>Straight line AB</td>
</tr>
<tr>
<td>m (L)</td>
<td>Measure of angle L</td>
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<td>~</td>
<td>Similarity</td>
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<td>&lt;</td>
<td>Less than</td>
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<tr>
<td>≤</td>
<td>Less than or equal to</td>
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<tr>
<td>&gt;</td>
<td>Greater than</td>
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<tr>
<td>≥</td>
<td>Greater than or equal to</td>
</tr>
<tr>
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</tr>
<tr>
<td>≡</td>
<td>is congruent to</td>
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Real Numbers

- Irrational Numbers
- Rational Numbers
  - Non-Integer Numbers
  - Integer Numbers
    - Natural Numbers
      - Positive Integer Numbers (Counting Numbers)
      - Zero
    - Negative Integer Numbers
Revision

Think and Discuss

The sets of numbers

The set of Counting numbers
\[ = \{1, 2, 3, \ldots\} \]

The set of Natural numbers
\[ N = \{0, 1, 2, 3, \ldots\} = \text{counting numbers} \cup \{0\} \]

The set of Integers
\[ Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} \]

The set of Positive integers
\[ Z^+ = \{1, 2, 3, \ldots\} = \text{Counting numbers} \]

The set of Negative integers
\[ Z^- = \{-1, -2, -3, \ldots\} \]

\[ Z = Z^+ \cup \{0\} \cup Z^- \]

The set of Rational numbers
\[ Q = \left\{ \frac{a}{b} : a, b \in Z, b \neq 0 \right\} \]

The absolute value of a rational number:

\[ | -7 | = 7, \quad |3| = 3, \quad |0| = 0, \quad \left| -\frac{5}{3} \right| = \frac{5}{3} \]

If \( |a| = 5 \) then \( a = \pm 5 \)
The Standard form of a rational number is:

\[ a \times 10^n \text{ where } n \in \mathbb{Z}, \ 1 \leq |a| < 10 \]

**For example:** The standard form of the number \( 25.32 \times 10^4 \)
\[ = 2.532 \times 10^5 \]
- The standard form of the number \( 0.00053 = 5.3 \times 10^{-4} \)

**The perfect square rational number:**

It is any positive number which can be written in the form of a square rational number i.e (rational number)^2

**Example** 1, 4, 25, \( \frac{9}{16} \), \( \frac{1}{4} \), ...

**The perfect cube of rational number:**

It is any rational number which can be written in the form of a cube rational number.

i.e (rational number)^3

**Example** 1, 8, -27, -216, \( \frac{8}{125} \), ...

**The square root of a perfect square rational number**

- The square root of the positive rational number \( a \) is that number whose square is equal to \( a \).
- \( \sqrt{0} = 0 \) the square root of zero is zero.
- Every perfect square rational number \( a \) has two square roots each one of them is an additive inverse to the other i.e. \( \sqrt{a}, -\sqrt{a} \)

**Example** \( \frac{16}{25} \) has two square roots: \( \frac{4}{5}, -\frac{4}{5} \)

- \( \sqrt{9} \) means the positive square root of 9 which is equal to 3

\[ \sqrt{\left( \frac{a}{b} \right)^2} = \left| \frac{a}{b} \right| \text{ i.e} \]

\[ \sqrt{(-7)^2} = |-7| = 7 \]

**Practice**

Complete the following table

<table>
<thead>
<tr>
<th>Number</th>
<th>Natural Number</th>
<th>Integer</th>
<th>Rational Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>-3</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>( \frac{3}{5} )</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>( \sqrt{\frac{9}{16}} )</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>(</td>
<td>5 - 7</td>
<td>)</td>
<td>No</td>
</tr>
</tbody>
</table>
Revision exercises

1. **Complete** Write the following numbers in the form \( \frac{a}{b} \) where \( a \) and \( b \) are two integers and there are \( n \)’t common factors between them, \( b \neq \text{zero} \).

   A. \( 0.2 = \) ............  
   B. \( 0.3 = \) ............  
   C. \( 25\% = \) ............  
   D. \( |-0.75| = \) ............  
   E. \( -6 = \) ............  
   F. \( 1 \frac{1}{4} = \) ............

2. **Choose** the correct answer:

   A. The solution set of the equation \( x + 5 = |-5| \) in \( \mathbb{N} \) is ............ \( \{0\} \) or \( \{10\} \) or \( \{-10\}, \varnothing \)
   B. The rational number lies between \( \frac{1}{5}, \frac{2}{5} \) is ............ \( \frac{2}{10} \) or \( \frac{1}{10} \) or \( 0.3 \) or \( -0.3 \)
   C. The product of the rational number \( \frac{a}{b} \) by its additive inverse is .......
   D. \( |-2| + |-4| + |6| = \) ............
   E. \( \sqrt{a^2} = \) ............
   (zero, \(- \frac{a}{b}\) or \( \frac{a^2}{b^2} \) or \( -\frac{a^3}{b^2} \))
   (zero, \(-|12|\), \(-12\), \(6\))
   (a or \(-a\) or \(|a|\) or \( \pm a\))

3. **Find** the value of \( x \) which satisfies each of the following equations. Determine whether this value is a natural, real or rational number.

   A. \( 5x + 3 = 20 \)
   B. \( 7x + 11 = 12 \)
   C. \( 3x + 5 = 1 \)
   D. \( x + 3 = 7 \)

4. **Find** in the simplest form:

   A. \( \sqrt{25 + 144} = \) ............
   B. The standard form of \( \frac{0}{00015} \) is ............
   C. \( \sqrt{0.16} + |-0.6| = \) ............
   D. \( 2^{0} + 2^{1} + 2^{2} + 2^{3} = \) ............
   E. The sum of the two square roots of the number \( 2 \frac{1}{4} = \) ............
   F. \( \sqrt{0.25} = \) ............
The cube root of a rational number

Think and Discuss

You have learned that:
The volume of a cube = the length of its side × itself × itself

Complete

The volume of the cube whose side length is equal to 7 cm = .......... × .......... × .......... = .......... cm³

Let's think

If we have a cube of volume 125 cm³, what is the length of its side?
We search for any three equal numbers of a product equal to 125. Then the number 125 can be factorized into its prime factors
125 = 5 × 5 × 5

∴ the cube of volume 125 cm³ has a side length = 5 cm
Therefore, 5 is called the cube root of 125 and it is written as \( \sqrt[3]{125} = 5 \).

You will learn how

- To find the cube root of a rational number using factorization.
- To find the cube root of a rational number using the calculator.
- To solve equations that include finding the cube root.
- To solve applications on the cube root of a rational number.

Key terms

- Cube root.

The cube root of the rational number a is that number whose cube is equal to a

- The cube root for the rational number a is symbolized by \( \sqrt[3]{a} \)
- The cube root for a positive rational number is also positive
  Ex: \( \sqrt[3]{125} = 5 \)
- The cube root for a negative rational number is also negative. Ex: \( \sqrt[3]{-8} = -2 \) why?
  \( \sqrt[3]{0} = 0 \)
  \( \sqrt[3]{a^3} = a \)
To find the cube root of a perfect cube rational number:

- The number can be factorized into its prime factors.
- A calculator can be used.

**Remark** The perfect cube rational number has one cube root which is also a rational number, why?

**Examples**

1. Use factorization to find the value of each \( \sqrt[3]{1000} , \sqrt[3]{-216} , \sqrt[3]{\frac{27}{8}} \); then check your answer using the calculator.

**Solution**

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<table>
<thead>
<tr>
<th></th>
<th></th>
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<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>500</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>250</td>
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<td>5</td>
<td></td>
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<tr>
<td>1</td>
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</table>

\( \sqrt[3]{1000} = 5 \times 2 = 10 \)

\( \sqrt[3]{-216} = -2 \times 3 = -6 \)

\( \sqrt[3]{\frac{27}{8}} = \frac{3}{2} \)

Use your calculator to check your answer by pressing on \( \sqrt[3]{} \).

2. Find the length of the radius of a sphere whose volume is equal to \( 4851 \text{cm}^3 \) \((\pi = \frac{22}{7})\)

**Solution**

\[
\text{The volume of the sphere} = \frac{4}{3} \pi r^3
\]

\[
4851 = \frac{4}{3} \times \frac{22}{7} \times r^3
\]

\[
r^3 = \frac{4851 \times 3 \times 7}{4 \times 22} = \frac{9261}{8}
\]

\[
\therefore r^3 = \frac{3^3 \times 7^3}{2^3}
\]

\[
\therefore r = \frac{3 \times 7}{2}
\]

The volume of a sphere is \( \frac{4}{3} \pi r^3 \) where \( r \) is the length of the radius and \( \pi \) is known as the approximate ratio.
$r = \frac{3 \times 7}{2} = \frac{21}{2} = 10.5 \text{ cm}$

we can use the calculator to find $\sqrt[3]{\frac{9261}{8}}$ directly.

**Practice**

Find the diameter of the sphere whose volume is $113.04 \text{ cm}^3$ ($\pi = 3.14$)

**Example**

Solve each of the following equations in $Q$.

A. $x^3 = 8$
   
   B. $x^3 + 9 = 8$

   C. $(x - 2)^3 = 125$

   D. $(2x - 1)^3 - 10 = 54$

**Solution**

A. $x^3 = 8$
   
   $x = \frac{3}{\sqrt[3]{8}} = 2$

   $\therefore$ Solution set = \{2\}

B. $x^3 + 9 = 8$
   
   $x^3 = 8 - 9$
   
   $x^3 = -1$
   
   $x = \sqrt[3]{-1} = -1$

   $\therefore$ Solution set = \{-1\}

C. $(x - 2)^3 = 125$
   
   $x - 2 = \sqrt[3]{125}$
   
   $x - 2 = 5$
   
   $x = 7$

   $\therefore$ Solution set = \{7\}

D. $(2x - 1)^3 - 10 = 54$
   
   $(2x - 1)^3 = 64$
   
   $2x - 1 = \sqrt[3]{64}$
   
   $2x - 1 = 4$
   
   $2x = 5$
   
   $x = \frac{5}{2}$

   $\therefore$ Solution set = \{\frac{5}{2}\}

**Practice**

Solve the following equations in $Q$: $(x + 1)^3 = 27$, $(x + 1)^3 = -27$
Exercises (1-1)

1. Fill in the following table:

<table>
<thead>
<tr>
<th>Number a</th>
<th>8</th>
<th>125</th>
<th>-27</th>
<th>..........</th>
<th>3 3/8</th>
<th>-8/125</th>
<th>..........</th>
<th>..........</th>
</tr>
</thead>
<tbody>
<tr>
<td>³√a</td>
<td></td>
<td></td>
<td></td>
<td>-10</td>
<td></td>
<td></td>
<td>6</td>
<td>-4</td>
</tr>
</tbody>
</table>

2. Complete
   
   A ³√-125 = .......... 
   B ³√343 = .......... 
   C ³√8 + ³√-8 = .......... 
   D ³√0.001 = .......... 
   E ³√27 - ³√64 = .......... 
   F ³√a³ = .......... 

3. Choose the correct answer from the Parentheses:
   
   A ³√(-8)² = .......... (2 or -2 or 4 or -4)
   B ³√25 - ³√-125 = .......... (10 or 0 or 5 or ±5)
   C ³√3 ³/₈ + ³√0.25 = .......... (3/2 or 1/2 or 2 or -2)
   D ³√1000 × ³√-0.008 = .......... (1/2 or 10 or 2 or -2)
   E The lateral area of a cube whose volume is 216 cm³ = .......... cm² (36 or 6 or 144 or 216)
   F ³√x⁶ = .......... (x³ or x² or x or x⁴)
   G ³√-27 + ³√12 ³/₄ + ³√0.125 = .......... (1 or 0 or -1 or 11/2)

4. Find the value of x in each of following cases:
   
   A √x = 5
   B ³√x = -1/2
   C ³√x = - √4
   D x³ = -8
   E x³ - 125 = 0
   F x² = 64

5. Find the solution set for each of the following equations in Q:
   
   A x³ + 27 = 0
   B 8x³ + 7 = 8
   C (x + 3)³ = 343
   D (5x - 2)³ + 10 = 18

6. Application problems
   
   A Find the side length of a cube vessel with capacity of one liter.
   B Find the diameter length of a sphere whose volume = \(\frac{1372}{81}\) \(\pi\) cubic unit (Use \(\pi = \frac{22}{7}\)) (volume of a sphere = \(\frac{4}{3} \pi r³\))
The set of Irrational numbers Q`

Think and Discuss

you have learned that: A rational number is that number which can be put in the form:

$$\frac{a}{b}$$ : where $$a, b \in \mathbb{Z}, b \neq 0$$

for example: when solving the equation $$4x^2 = 25$$

then $$x^2 = \frac{25}{4}$$ : $$x = \pm \frac{5}{2}$$

Remark Each of $$\frac{5}{2}, -\frac{5}{2}$$ is a rational number.

However, there are many numbers which cannot be put in the form $$\frac{a}{b}$$ where $$a, b \in \mathbb{Z}, b \neq 0$$

for example: when solving the equation $$x^2 = 2$$, we can not find any rational number whose square is equal to 2

The irrational number

It is that number which cannot be put in the form $$\frac{a}{b}$$ where $$a, b \in \mathbb{Z}, b \neq 0$$

the following are examples to irrational numbers.

First: the square roots of the positive numbers which are not perfect squares

Ex : $$\sqrt{2}, \sqrt{5}, -\sqrt{6}, \sqrt{7}$$

Second: the cube roots of those numbers that are not perfect cubes

Ex : $$\sqrt[3]{4}, \sqrt[3]{2}, \sqrt[3]{11}, ...$$

Third: the pi $$\pi$$ (the approximation ratio)

Where it is impossible to find any exact value for any of the previous number. why?
Those numbers and others form a set which is called the set of irrational numbers which is denoted by the symbol \( \mathbb{Q}^* \).

\[ \mathbb{Q} \cap \mathbb{Q}^* = \emptyset \]

**Think**: is \( \sqrt{-1} \) an irrational number? why?

### Exercises (1-2)

1. **Complete** using the symbol \( \mathbb{Q} \) or \( \mathbb{Q}^* \).
   - A. \( 5 \in \) .............
   - B. \( \sqrt{10} \in \) .............
   - C. \( 0 \in \) .............
   - D. \( -0.7 \in \) .............
   - E. \( \sqrt{8} \in \) .............
   - F. \( \sqrt{6} \in \) .............
   - G. \( \sqrt{-9} \in \) .............
   - H. \( \pi \in \) .............

2. **Put (✓) on the true sentence and (✗) on the false sentence**
   - A. \( 2.3 \times 10^5 \in \mathbb{Q} \ (\ ✔) \)
   - B. \( |-5| \in \mathbb{Q}^* \ (\ ✗) \)
   - C. \( \frac{\text{zero}}{5} \in \mathbb{Q} \ (\ ✔) \)
   - D. \( \sqrt{-4} \in \mathbb{Q}^* \ (\ ✗) \)
   - E. \( \sqrt{1000} \in \mathbb{Q} \ (\ ✗) \)
   - F. \( \sqrt{7} > 3 \ (\ ✔) \)
   - G. \( \sqrt{10} > 2 \ (\ ✗) \)
   - H. \( \sqrt{20} > \sqrt{9} \ (\ ✗) \)

The side length of a square whose surface area = 6cm² is a rational number (✗)

3. **Choose the correct answer**:
   - A. The surface area of a square whose side length is \( \sqrt{3} \) = ...........
     - \( (4 \sqrt{3} \) or 9 or 3 or 6)
   - B. The irrational number lies between 3 and 4 is ...........
     - \( (3.5 \) or \( \frac{1}{8} \) or \( \sqrt{13} \) or \( \sqrt{20} \))
   - C. The irrational number lies between -2 and -1 is ...........
     - \( (-3 \) or \( -1 \frac{1}{2} \) or \( -\sqrt{3} \) or \( \sqrt{2} \))
Finding the approximate value of an Irrational number

Think and discuss

Can you find the two rational numbers which the irrational number $\sqrt{2}$ is located between them.

**Remark**

$\sqrt{2}$ is between $\sqrt{1}$, $\sqrt{4}$ i.e $1 < \sqrt{2} < 2$

i.e. $\sqrt{2} = 1 + a$ decimal fraction

To find the approximate value of $\sqrt{2}$. We check the values of the following numbers:

$(1.1)^2 = 1.21$, $(1.2)^2 = 1.44$, $(1.3)^2 = 1.69$, $(1.4)^2 = 1.96$, $(1.5)^2 = 2.25$

$\therefore 1.96 < 2 < 2.25$

$\therefore 1.4 < \sqrt{2} < 1.5$

i.e. $\sqrt{2} = 1.4 + a$ decimal fraction

i.e. $1.41 < \sqrt{2} < 1.42$

Use the calculator to check you answer.

Representing the irrational number on the number line.

How can the point represents $\sqrt{2}$ be located on the number line?

If we draw the right triangle ABC at B which is an isosceles triangle also.

where $AB = BC = \text{one unit of length}$

Then $(AC)^2 = (AB)^2 + (BC)^2 = 1^2 + 1^2 = 2$

$\therefore AC = \sqrt{2}$ unit of length.
o draw the number line and place the sharp point of the compasses at point O, then
adjust the compasses to a length that is equal to $\overline{AC}$ and draw an arc that intersects the
number line on the right of $O$ and at the point $X$, where that point represents $\sqrt{2}$

o Using the same length, we can label the point $X^*$ which represent $-\sqrt{2}$ where $X^*$ is
on the left of the point $O$.

Think : Label the point which represents $3 + \sqrt{2}$ on the number line.

Activity : Draw the square $OABC$ whose side length is equal to one unit of length.

The length of its diagonal $= \sqrt{1+1} = \sqrt{2}$ unit of length

\[ \therefore \overline{OB} = \sqrt{2} \]

o Place the sharp point of the compasses at point $O$ and draw a semi-circle whose di-
amerter $= \text{the length of } \overline{OB} = \sqrt{2}$.

o $\overrightarrow{OA} \cap \text{the semi-circle} = \{X, X^*\}$ where $X$ represents the number $\sqrt{2}$, $x^*$ represents
the number $-\sqrt{2}$.

o Draw $\overline{XD} // \overline{AB}$ and intersects $\overline{CB}$ at $D$

\[ (OD)^2 = (OX)^2 + (XD)^2 = (\sqrt{2})^2 + (1)^2 = 3 \]

\[ \therefore \overline{OD} = \sqrt{3} \]

o Place the sharp point of the compasses at point $O$ and adjust it to a length which
is equal to the length of $\overline{OD}$, then draw semi-circle that intersects with $\overrightarrow{OA}$ at
points $Y, Y^*$

\[ \therefore \overline{OY} = \sqrt{3} \text{ i.e. point Y represents } \sqrt{3}, \text{ while point Y^* represents } -\sqrt{3} \]

o Continue using the same method to represent $\sqrt{4}, \sqrt{5}, \sqrt{6}, ...$

also $-\sqrt{4}, -\sqrt{5}, -\sqrt{6}, ...$
Practice

1. Find:
   A. Two consecutive integers that \( \sqrt{5} \) lies between them.
   B. Two consecutive integers that \( \sqrt{12} \) lies between them.
   C. Two consecutive integers that \( \sqrt{10} \) lies between them.
   D. Two consecutive integers that \( \sqrt{-20} \) lies between them.

2. Prove that:
   A. \( \sqrt{3} \) lies between 1.7, 1.8.
   B. \( \sqrt{15} \) lies between 2.4, 2.5.

3. Find the value of \( \sqrt{11} \) to the nearest hundredth.
4. Find the value of \( \sqrt{2} \) to the nearest tenth.
5. Draw the number line and label the point which represents the irrational number \( \sqrt{3} \).
6. Draw the number line and label the point which represents the irrational number \( 1 + \sqrt{2} \).

Example (1)

Find the solutions set for each of the following equations in \( \mathbb{Q}^* \):

A. \( x^2 = 2 \)
   Solution: \( x = \pm \sqrt{2} \)  Solution set = \{ -\sqrt{2}, \sqrt{2} \}

B. \( x^3 = 5 \)
   Solution: \( x = \sqrt[3]{5} \)  Solution set = \{ \sqrt[3]{5} \}

C. \( \frac{4}{3} x^2 = 1 \)
   \( \frac{3}{4} \times \frac{4}{3} x^2 = \frac{3}{4} \times 1 \)
   \( x^2 = \frac{3}{4} \)
   \( x = \pm \frac{\sqrt{3}}{2} = \pm \sqrt{\frac{3}{4}} = \pm \frac{\sqrt{3}}{2} \)  Solution set = \{ -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2} \}
0.001 \times 3 = -8
\therefore x^3 = -\frac{8}{0.001} = -8000
\therefore x = \sqrt[3]{-8000} = -20 \in \mathbb{Q}
The solution set in \mathbb{Q} = \emptyset

Example (2)

Find the length of each of the side and the diagonal of a square whose area is 7 cm².

Solution

Let the length of the side be \( x \) cm,
then the area = \( x \times x = x^2 \)
Where \( L \) is the square diagonal length
\( x^2 = 7 \)
\( \therefore x = \pm \sqrt{7} \) cm
\( \therefore x = \sqrt{7} \) cm why?
To find the diagonal of the square: use pythagorean theorem
\( L^2 = x^2 + x^2 \) Where \( L \) is the square diagonal length.
\( \therefore L^2 = 14 \)
\( \therefore L = \pm \sqrt{14} \) cm
\( \therefore L = \sqrt{14} \) cm why?

Example (3)

Find: the circumference of a circle whose area is 3 \( \pi \) cm²

Solution

The area of the circle = \( \pi r^2 \)
\( 3 \pi = \pi r^2 \)
\( \therefore r^2 = 3 \)
\( r = \sqrt{3} \) cm or \( r = -\sqrt{3} \) cm (refused)
the circumference = \( 2 \pi r = 2 \pi \times \sqrt{3} = 2 \sqrt{3} \pi \) cm.
Exercises (1-3)

1. Circle the irrational number in each of the following:
   \[ \sqrt{3}, -0.2, \sqrt{-1}, 0, \sqrt[9]{9}, -\sqrt{\frac{4}{25}} \]

2. **Find** the value of \( x \) in each of the following cases and determine whether \( x \in \mathbb{Q} \) or \( x \notin \mathbb{Q} \):
   
   - A. \( 4x^2 = 9 \)
   - B. \( 2x^2 = 6 \)
   - C. \( x^3 = 125 \)
   - D. \( x^3 = 10 \)
   - E. \( (x - 1)^2 = 4 \)
   - F. \( (x - 2)^3 = 1 \)

3. **Find** the approximate value for \( \sqrt{10} \), and check your answer using the calculator.

4. **Think**: If \( x \) is an integer, find the value of \( x \) in each of the following cases:
   
   - A. \( x < \sqrt{7} < x + 1 \)
   - B. \( x < \sqrt{80} < x + 1 \)
   - C. \( x < \sqrt{125} < x + 1 \)
   - D. \( x < \sqrt{5} < x + 1 \)
   - E. \( x < \sqrt{30} < x + 1 \)
   - F. \( x < \sqrt{100} < x + 1 \)

5. **Choose** the correct answer:
   
   - A. the irrational number lies between 2 and 3 is. \( (\sqrt{10} \text{ or } \sqrt{7} \text{ or } 2.5 \text{ or } \sqrt{3}) \).
   - B. \( \sqrt{10} \approx \frac{10}{3} \) (2.99 or 3.71 or 3 or -3.2).
   - C. The nearest integer to \( \sqrt{25} \) is \( \frac{1}{2} \) (5 or 3 or 2 or 12.5).
   - D. The square whose area is 10 cm², its side length is \( \frac{1}{2} \) cm
     \( \left(5 \text{ or } -5 \text{ or } \sqrt{10} \text{ or } -\sqrt{10}\right)\).
   - E. The cube whose volume is 64 cm³ its side length is \( \frac{1}{2} \) cm (8 or 4 or 16 or 64).

6. Draw the number line and label point A which represents \( \sqrt{2} \) and
   Label point B which represents \( 1 + \sqrt{2} \)
   Label point C which represents \( 1 - \sqrt{2} \)

7. Draw the right triangle ABC at B where \( AB = 2 \text{ cm}, BC = 3 \text{ cm}, \) then use the figure to label a point which represents \( \sqrt{13} \) and a point which represents \( -\sqrt{13} \) on the number line.
Think and Discuss

You have learned the set of rational numbers \( (Q) \), you have also found that there are other numbers that form the set of irrational number \( Q' \) such as \( \sqrt{2}, \sqrt{2}, \pi, ... \). However, the union of these two sets forms a new set called the set of the real numbers, and it is denoted by the symbol \( R \)

\[ R = Q \cup Q' \]

Look at the opposite Venn diagram, you find that:

1. \( R = Q \cup Q' \)
2. Any natural, integer, rational or irrational number is a real number

\[ N \subset Z \subset Q \subset R \quad \text{and so is} \quad Q' \subset R \]

Think Give examples from your own to some real numbers which are rational or irrational numbers.

3. Every real number is represented by one point on the number line.

First: zero is represented by the origin \( O \).
Second: the positive real numbers are represented by all the points on the number line that are located on the right side of \( O \)
Third: the negative real numbers are represented by all the points on the number line that are located on the left side of \( O \)
Practice

1. Put each of the following numbers in its suitable place on the opposite venn diagram.
   \( \frac{1}{2}, -4, 9, \sqrt{5}, 0, 6, \frac{7}{9}, \sqrt{-2}, \sqrt{16}, 0, 5 \)

2. Label point A on the number line which represents \( \sqrt{-8} \), and point B which represents \( \sqrt{9} \), then find the length of \( AB \).

3. State if each sentence is true or false:
   
   A. Every natural number is a positive real number.
   
   B. Every integer is a real number.

   Remark
   \( \sqrt{-1} = -1 \) because \(-1 \times -1 \times -1 = -1\)
   While \( \sqrt{-1} \notin R \) because there is no real number if multiplied by itself, the product is -1.

Discuss with your teacher and classmates: Are there any non-Real number?
Exercises (1 - 4)

1. Study the previous chart and answer the following by placing (√) on the true sentence and (✗) on the false sentence:
   - A. Every natural number is an integer. ( ✓ )
   - B. Zero ∈ The set of rational numbers ( ✓ )
   - C. Z = Z⁺ ∪ Z⁻ ( ✓ )
   - D. Any non-integer number is a rational number ( ✗ )

2. Complete the following table by placing (√) in the suitable place as shown in the first case:

<table>
<thead>
<tr>
<th>Number</th>
<th>natural number</th>
<th>integer number</th>
<th>rational number</th>
<th>irrational number</th>
<th>real number</th>
</tr>
</thead>
<tbody>
<tr>
<td>-5</td>
<td>✗</td>
<td>✓</td>
<td>✓</td>
<td>✗</td>
<td>✓</td>
</tr>
<tr>
<td>√2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>✓</td>
</tr>
<tr>
<td>1 1/2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>√9</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>-2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>- √4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5/2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0,3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>√-1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Ordering numbers at $\mathbb{R}$

**Think and Discuss**

If $A$, $B$ are two points that belong to the straight line $L$, and we determined a certain direction as shown by the arrow; then we can say that:

- The point $B$ follows the point $A$. i.e on its right hand side.
- The point $A$ precedes the point $B$. i.e on its left hand side.

The same applies for all the points on the straight line.
However, If we know that every point on the straight line represent a real number. We can say that:

**The set of real number is an ordered set.**

**The properties of order:**

1. If $x$, $y$ are two real numbers represented on the number line by the two points $A$, $B$ respectively, the ordering relation can be one of the following three cases:

   - $x = y$  
     
     ![Diagram showing $x = y$](image)

   - $x > y$
     
     ![Diagram showing $x > y$](image)

   - $x < y$
     
     ![Diagram showing $x < y$](image)

   - $A$ is congruent to $B$ so $x = y$
   - $A$ follows $B$ so $x > y$
   - $A$ precedes $B$ so $x < y$

2. If $x$ is a real number represented by the point $A$ on the number line while $O$ is the origin point which represents the zero, then the ordering relation can be one of the following three cases:

   - $x = 0$
     
     ![Diagram showing $x = 0$](image)

   - $x > 0$
     
     ![Diagram showing $x > 0$](image)

   - $x < 0$
     
     ![Diagram showing $x < 0$](image)

   - $A$ is congruent to $O$  \(\therefore x = 0\)
   - $A$ is on the right of $O$  \(\therefore x > 0\) then $x$ is a positive real number.
   - $A$ is on the left of $O$  \(\therefore x < 0\) then $x$ is a negative real number.

**You will learn how**

- To define the ordering relation in $\mathbb{R}$.

**Key terms**

- Ordering relation
- More than
- Less than
- Equal to
- Ascending order
- Descending order
The set of the positive real numbers: \( R^+ = \{x : x \in \mathbb{R}, \ x > 0\} \)
The set of the negative real numbers: \( R^- = \{x : x \in \mathbb{R}, \ x < 0\} \)

\[ R = R^+ \cup \{0\} \cup R^- \]

Remark:
The set of non-negative real numbers = \( R^+ \cup \{0\} = \{x : x \geq 0, \ x \in \mathbb{R}\} \)
The set of the non-positive real numbers = \( R^- \cup \{0\} = \{x : x \leq 0, \ x \in \mathbb{R}\} \)

Example:

Arrange the following numbers ascendingly: \( \sqrt{27}, \ -\sqrt{45}, \ \sqrt{20}, \ 6, \ 0, \ \sqrt{-1} \)

Solution

\[ 6 = \sqrt{36}, \ \sqrt{-1} = -1 = -\sqrt{1} \]

The ascending order is from the smallest to the greatest.

\[ -\sqrt{45}, \ -\sqrt{1}, \ 0, \ \sqrt{20}, \ \sqrt{27}, \ \sqrt{36} \]

i.e. \( -\sqrt{45}, \ \sqrt{-1}, \ 0, \ \sqrt{20}, \ \sqrt{27}, \ 6 \).

Exercises (1-5)

1. Arrange the following number descendingly: \( \sqrt{62}, \ 8, \ -\sqrt{50}, \ \sqrt{70} \)

2. If \( x \in \mathbb{R} \), state whether \( x \) is a positive or negative or anything else in each of the following.
   - A \( x > 0 \)
   - B \( x < 0 \)
   - C \( x > |\ -5| \)

3. Prove that \( \sqrt{3} \) lies between 1.7, 1.8, then represent \( \sqrt{3} \), 1.7 and 1.8 on the number line.

4. Find the side length in a square whose area is 5 cm\(^2\). Is the side length a rational number?

5. Find the side length in a cube whose volume is 1.728 cm\(^3\). Is the side length a rational number?

6. Put the suitable notation (\( > \) or \( < \) or \( = \))
   - A \( \sqrt{5} \quad \ldots \quad 2 \)
   - B \( \sqrt{7} \quad \ldots \quad 2,6 \)
   - C \( \sqrt{24} \quad \ldots \quad -2 \)
   - D \( 1 + \sqrt{2} \quad \ldots \quad \sqrt{3} \)
   - E \( \sqrt{8} \quad \ldots \quad \sqrt{4} \)
   - F \( 3 - \sqrt{5} \quad \ldots \quad \sqrt{-1} \)
**Interval** is a subset of the set of real numbers

First: the limited intervals

If \( a, b \in \mathbb{R}, a < b \), then we can define each of:

**The closed interval**

\[ [a, b] = \{ x : a \leq x \leq b, \ x \in \mathbb{R} \} \]

\([a, b] \subset \mathbb{R}\) in which the elements are \( a, b \) and all the real numbers between them.

When we draw that interval, we put a shaded circle at each of the two points \( a \) and \( b \) then, we shade that area between them on the number line.

**The open interval**

\( ]a, b[ = \{ x : a < x < b, \ x \in \mathbb{R} \} \)

\( ]a, b[ \subset \mathbb{R}\) in which the elements are all the real numbers between the two numbers \( a, b \).

When we draw that interval, we put an unshaded circle at each of the two points which represent the two numbers \( a \) and \( b \) then, we shade that area between them on the number line.

**You will learn how**

- To define limited intervals.
- To define unlimited intervals.
- To recognize the operations on intervals.

**Key terms**

- Limited interval
- Closed interval
- Open interval
- Half-open interval
- Unlimited interval
- Union
- Intersection
- Difference
- Complement

**Practice**

Write down each of \([3,5], ]3,5[\) using the description method then represent them on the number line.
**Half open or (half closed) intervals**

- **[a, b]**
  - \([a, b] = \{x : a \leq x < b, x \in \mathbb{R}\}\)
  - \([a, b] \subset \mathbb{R}\) where its elements are the number \(a\) and all the numbers between \(a\) and \(b\).

- **]a, b]**
  - \([a, b] = \{x : a < x \leq b, x \in \mathbb{R}\}\)
  - \([a, b] \subset \mathbb{R}\) where its elements are the number \(b\) and all the number between \(a\) and \(b\).

---

**Practice**

Write down each of the two intervals: [3, 5[, ]3, 5] using the description method, then represent them on the number line.

**Examples:**

Represent each of the following intervals on the number line: [-1, 4], ]-1, 4[, ]-1, 4], {-1, 4}

**Solution**

- **[−1, 4]**
  - Closed interval
  - \([-1, 4]\)

- **]-1, 4[**
  - Open interval
  - \((−1, 4)\)

- **]-1, 4]**
  - Half-open interval
  - \([−1, 4)\)

- **{-1, 4}**
  - Set
  - \(\{−1, 4\}\)

Discuss with your teacher and your classmates whether the interval is a finite or an infinite set.

**Practice**

1. Write down the following sets in the form of intervals, then represent them on the number line:
   - \(A\) \(X = \{x : 2 < x < 5, x \in \mathbb{R}\}\)
   - \(B\) \(X = \{x : -2 \leq x < 3, x \in \mathbb{R}\}\)
   - \(C\) \(X = \{x : 0 \leq x \leq 4, x \in \mathbb{R}\}\)
   - \(D\) \(X = \{x : -3 < x \leq -1, x \in \mathbb{R}\}\)
4. **Put** The suitable symbol ∈ or ∉ to make each sentence true.

   A. 3        B. -2        C. $\frac{1}{2}$
   [-1, 3]    [1, 3]     [0, 1]

   D. $\sqrt{2}$     E. 4        F. $\sqrt{-8}$
   [1, 2]     [0, 5]     [-1, 2]

   G. |-5|        H. $2.3 \times 10^{-5}$
   [4, 6]     [0, 1]

3. **Write** down the interval represented by each of the following figures:

   - **A**: (0, 3)
   - **B**: (-4, 1)
   - **C**: (-3, 1)
   - **D**: (0, 6)

**Second: The unlimited intervals**

**You know that**: If the number line of real numbers is expanded on its two directions, we get more positive real numbers at the right direction and more negative real numbers at the left direction such all those numbers are located on that line.

- The symbol ($\infty$) is read (infinity) and it is more than any imagined real number, $\infty \notin R$
- The symbol ($-\infty$) is read (negative infinity) and it is less then any imagined real number, $-\infty \notin R$
- The two symbols $\infty$, $-\infty$ can not be represented by any points on the number line and they are expansions to the number line at its two directions.

**If $a$ is a real number, then we can define the following unlimited intervals:**

- The interval $[a, \infty]$
  $[a, \infty] = \{ x : x \geq a, x \in R \}$
  - That interval represents the number $a$ and all the real numbers which are more than $a$

- The interval $[-\infty, a]$
  $[-\infty, a] = \{ x : x \leq a, x \in R \}$
  - That interval represents the number $a$ and all the real number which are less than $a$.

**Write down** each of the following intervals $[3, \infty]$, $[-\infty, 3]$ using the description method, then represent them on the number line.
the interval \([a, \infty[\]
\([a, \infty[ = \{x : x > a, \ x \in \mathbb{R}\}

That interval represents all the real numbers which are more than a

the interval \([-\infty, a]\]
\([-\infty, a[ = \{x : x < a, \ x \in \mathbb{R}\}

that interval represents all the real numbers which are less than a

Write down the two intervals \([3, \infty[ , \ [-\infty, 3]\] using the description method, then represent them on the number line

Remark:
The set of real numbers (\(\mathbb{R}\)) can be represented in the form of the interval \([-\infty, \infty[\)

The set of the positive real numbers \(\mathbb{R}^+ = ]0, \infty[\)
The set of the negative real numbers \(\mathbb{R}^- = [-\infty, 0[\)
The set of non-negative real numbers = \([0, \infty[\)
The set of non-positive real numbers = \([-\infty, 0]\)

Practice

1. Write down the following sets in the form of intervals, then represent them on the number line.
   - A: \(X = \{x : x \geq 2, \ x \in \mathbb{R}\}\)
   - B: \(X = \{x : x < 3, \ x \in \mathbb{R}\}\)
   - C: \(X = \{x : x > -7, \ x \in \mathbb{R}\}\)
   - D: \(X = \{x : x \leq \sqrt{-8}, \ x \in \mathbb{R}\}\)
   - E: \(X = \text{the set of all the real numbers more than } \mid -3 \mid\)

2. Put the suitable symbol \(\in\) or \(\notin\) or \(\leq\) or \(\geq\) to make each statement true:
   - A: \(3 \quad \text{.........] -}\infty, 4[\) B: \([1, 2] \quad \text{.........] -}\infty, -1, \infty[\)
   - C: \((-5) \quad \text{.........] -}\infty, -6[\) D: \([0, 2[ \quad \text{.........] 0, \infty[\)
   - E: \(3 \times 10^{10} \quad \text{.........] 3, \infty[}\) G: \([-3, 1] \quad \text{.........] 2, \infty[\)

Mathematics - Second Preparatory
Operations on intervals

Since all the intervals are subsets of the set of the real number $R$, the operations of union, intersection, difference, and complement can be applied on the intervals. The graphical representation of the intervals on the number line contributes to determine and verify the result of any operation. This can be clarified from the following examples:

**Examples**

1. If $X = [-2, 3], Y = [1, 5]$, find the following using the number line:
   - $X \cap Y$
   - $X \cup Y$

   **Solution**
   - $X \cap Y = [-2, 3] \cap [1, 5] = [1, 3]$  
   - $X \cup Y = [-2, 3] \cup [1, 5] = [-2, 5]$

2. If $M = [2, \infty], J = [-2, 3]$, find the following using the number line:
   - $M \setminus J$
   - $M \cap J$
   - $M \cup J$
   - $J \cup \{2, 3\}$
   - $M^c$
   - $J^c$

   **Solution**
   - $M \setminus J = [2, \infty] \setminus [-2, 3] = [3, \infty]$
   - $M \cap J = [2, \infty] \cap [-2, 3] = [2, 3]$
   - $M \cup J = [2, \infty] \cup [-2, 3] = [-2, \infty]$
   - $J \cup \{2, 3\} = [-2, 3] \cup \{2, 3\} = [-2, 3]$
   - $M^c = \mathbb{R} \setminus [2, \infty] = (-\infty, 2]$
   - $J^c = \mathbb{R} \setminus [-2, 3] = (-\infty, -2] \cup [3, \infty]$

**Practice**

Put (✓) on the true sentence and (✗) on the false sentence:

A. $[-2, 5] - \{2, 5\} = [-2, 5]$
B. $[-1, 3] \cup [-1, 0] = [-1, 0]$
C. $[2, 5] - \{5\} = [2, 5]$
D. $[-1, 3] \cap \{1, 4\} = [1, 3]$
E. $[-2, 5] \cup [1, 5] = [-2, 5]$
F. $[5, \infty] \setminus (-\infty, 5] = [5, \infty]$
# Exercises (1-6)

1. **Complete** the following table as shown in the first example:

<table>
<thead>
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2. **Complete** using \( \in \) or \( \notin \):

   - A \( 3 \ldots [2 , 3] \)
   - B \( \sqrt{-1} \ldots ]-\infty , 1[ \)
   - C \( 2 \ldots [1 , 7] \)
   - D \( \sqrt{9} \ldots ]-3 , \infty[ \)
   - E \( |-2| \ldots [2 , \infty[ \)
   - F \( 1.3 \times 10^{-5} \ldots \mathbb{R}^+ \)

3. **Choose** the correct answer:

   - A \([2 , 7] \setminus \{2 , 7\} = \ldots \) \( ([1 , 6] \text{ or } \emptyset \text{ or } ]2 , 7[ \text{ or } \emptyset) \)
   - B \([0 , 5] \cup [3 , 8[ = \ldots \) \( ([3 , 5] \text{ or } [3 , 5] \text{ or } [0 , 8] \text{ or } [0 , 8]) \)
   - C \([1 , 5] \cap ]-2 , 3] = \ldots \) \( ([1 , 3] \text{ or } ]1 , 3[ \text{ or } [1 , 3] \text{ or } [1 , 3]) \)
   - D \([-1 , 2[ \setminus [-1 , 4] = \ldots \) \( (]-1 , 1[ \text{ or } \{-1 , 1\} \text{ or } ]-1 , 1[ \text{ or } [-1 , 1]) \)

4. If \( X = [-1 , 4] \), \( Y = [3 , \infty[ \), \( Z = \{3 , 4\} \), find each of the following using the number line:

   - A \( X \cup Y \)
   - B \( X \cap Y \)
   - C \( X - Y \)
   - D \( X - Z \)
   - E \( Y \cap Z \)
   - F \( Y - X \)
   - G \( X^c \)
   - H \( Y^c \)
First: The properties of adding the real numbers:

You have determined the location of the point X which represents the number $1 + \sqrt{2}$ on the number line. Since it represents the sum of the two real numbers $1$ and $\sqrt{2}$, then the sum of every two real numbers is a real number.

I.e., the set of the real numbers $\mathbb{R}$ is closed under the operation of addition.

### The closure property

If $a \in \mathbb{R}$, $b \in \mathbb{R}$ then $(a + b) \in \mathbb{R}$

**For example:** each of $2 + 3$, $1 + \sqrt{2}$, $-2 + \sqrt{5}$ and $2 + \sqrt{3}$ are real numbers.

### The commutative property

If $a \in \mathbb{R}$, $b \in \mathbb{R}$ then $a + b = b + a$

**For example:** $2 + \sqrt{3} = \sqrt{3} + 2$, $3 - \sqrt{5} = -\sqrt{5} + 3$

### The associative property

If $a \in \mathbb{R}$, $b \in \mathbb{R}$, $c \in \mathbb{R}$, then $(a + b) + c = a + (b + c) = a + b + c$

**For example:** $(3 + \sqrt{2}) + 5 = 3 + (\sqrt{2} + 5)$ associative property

$= 3 + (5 + \sqrt{2})$ commutative property

$= 3 + 5 + \sqrt{2}$ associative property

$= 8 + \sqrt{2}$

You will learn how:

- To solve operations on the real numbers.
- To define the properties of operations on the real numbers.

**Key terms**

- Closure property.
- Commutative property.
- Associative property.
- Additive neutral.
- Additive inverse.
- Multiplicative neutral.
- Multiplicative inverse.
- Distribution of multiplication on addition or subtraction.
Zero is the additive neutral element: If \( a \in \mathbb{R} \) then \( a + 0 = 0 + a = a \)

For example: \( \sqrt{5} + 0 = 0 + \sqrt{5} = \sqrt{5} \), \( -\sqrt{4} + 0 = 0 + (-\sqrt{4}) = -\sqrt{4} \)

Each real number has an additive inverse For a number \( a \in \mathbb{R} \) there is \( (-a) \in \mathbb{R} \) where \( a + (-a) = (-a) + a = \) zero.

For example \( \sqrt{3} \in \mathbb{R} \), has additive inverse \( (-\sqrt{3}) \in \mathbb{R} \) where \( \sqrt{3} + (-\sqrt{3}) = (-\sqrt{3}) + \sqrt{3} = \) zero.

Practice

1. Complete the following to have a true sentence:
   - A  \( \sqrt{2} + 5 = 5 + \) ..............
   - B  \( \sqrt{11} + (-\sqrt{11}) = \) ..............
   - C  \( 7 + \sqrt{3} = 5 + (\) .............. + \) ..............\)
   - D  the additive inverse for \( \sqrt{8} \) is ..............
   - E  the additive inverse for \( (1 - \sqrt{2}) \) is ..............
   - F  \( \sqrt{3} + (-\sqrt{3}) = \) ..............
   - G  \( 7 + \sqrt{5} - 3 = \) ..............
   - H  \( (4 + \sqrt{7}) + (3 - \sqrt{7}) = \) ..............
   - I  If \( a \in \mathbb{R}, b \in \mathbb{R} \), then \( a - b \) means the sum of the number \( a \) and .............. of the number \( b \).
   - J  If \( a \in \mathbb{N}, b \in \mathbb{Q}, c \in \mathbb{R} \), then \( a + b + c \in \) ..............

2. Discuss the following with your teacher and classmates, then give examples:
   - A  Is subtraction a commutative operation in \( \mathbb{R} \)?
   - B  Is subtraction an associative operation in \( \mathbb{R} \)?
Second: The properties of multiplying the real numbers

**The closure property**
If \( a \in R, b \in R \) then \( a \times b \in R \)

the set of real number is closed under the operation of multiplication.

i.e the product of multiplying every two real number is a real number.

**for example:**
- \( 5 \times \sqrt{2} = 5 \sqrt{2} \in R, \sqrt{3} \times \sqrt{3} = 3 \in R \)
- \( -2 \times \sqrt{5} = -2 \sqrt{5} \in R, \frac{2}{3} \times \pi = \frac{2}{3} \pi \in R \)
- \( 2 \sqrt{3} \times \sqrt{3} = 6 \in R, 2 \sqrt{3} \times 5 = 10 \sqrt{3} \in R \)

**Commutative property**
If \( a \in R \) and \( b \in R \), then \( a \times b = b \times a \)

**for example:**
- \( \sqrt{2} \times 3 = 3 \times \sqrt{2} = 3 \sqrt{2} \)

**The associative property**
If \( a \in R, b \in R, c \in R \), then:

\[
(a \times b) \times c = a \times (b \times c) = a \times b \times c
\]

**for example:**
- \( \sqrt{2} \times (5 \times \sqrt{2}) = (\sqrt{2} \times 5) \times \sqrt{2} = (5 \times \sqrt{2}) \times \sqrt{2} \)
- \( = 5 \times \sqrt{2} \times \sqrt{2} = 5 \times 2 = 10 \)

**One is the multiplicative neutral**
If \( a \in R \), then \( a \times 1 = 1 \times a = a \)

**for example:**
- \( 2 \sqrt{5} \times 1 = 1 \times 2 \sqrt{5} = 2 \sqrt{5} \)

**Every real number \( \neq 0 \) has a multiplicative inverse**
If \( a \neq 0 \)

It exist an real number \( \frac{1}{a} \) such that

\[
a \times \frac{1}{a} = \frac{1}{a} \times a = 1 \quad (1 \text{ is the neutral element of multiplication})
\]

**for example:**
the multiplicative inverse for \( \frac{\sqrt{3}}{2} \) is \( \frac{2}{\sqrt{3}} \)

where \( \frac{\sqrt{3}}{2} \times \frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{2} = 1 \)

**Remark:**
\[
\frac{a}{b} = a \times \frac{1}{b} , \quad b \neq 0
\]

i.e. \( \frac{a}{b} = a \times \text{the multiplicative inverse of } b \).

Discuss with your teacher: is the division operation commutative in \( R \)? Is the division operation associative in \( R \)?
Examples

Write down each of the following numbers $\frac{6}{\sqrt{2}}$, $-\frac{5}{\sqrt{3}}$, $\frac{15}{2\sqrt{5}}$ where the denominator is an integer.

Solution

Note that the multiplicative neutral is 1 and it can be written in the form $\frac{\sqrt{2}}{\sqrt{2}}$ or $\frac{\sqrt{3}}{\sqrt{3}}$ or $\frac{\sqrt{5}}{\sqrt{5}}$ or ...

\[
\frac{6}{\sqrt{2}} = \frac{6}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{6\sqrt{2}}{2} = \frac{3\sqrt{2}}{1} = 3\sqrt{2}
\]

\[-\frac{5}{\sqrt{3}} = -\frac{5}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = -\frac{5\sqrt{3}}{3}
\]

\[\frac{15}{2\sqrt{5}} = \frac{15}{2\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{15\sqrt{5}}{2 \times 5} = \frac{3\sqrt{5}}{2}
\]

Practice

1. Complete the following to have a true sentence:
   
   A. $\sqrt{2} + \sqrt{2} + \sqrt{2} = \times \sqrt{2} =$
   B. $3 \times \sqrt{5} = \sqrt{5} \times$
   C. $\sqrt{7} \times \sqrt{7} =$
   D. $2\sqrt{5} \times 3\sqrt{5} =$
   E. The multiplicative neutral in R is the number
   F. The multiplicative inverse for $\frac{3}{\sqrt{2}}$ is

2. Write each of the following numbers such that the denominator is an integer:
   
   A. $\frac{15}{\sqrt{6}}$
   B. $\frac{8}{3\sqrt{2}}$
   C. $-\frac{6}{\sqrt{3}}$
   D. $\frac{25}{2\sqrt{10}}$

Distribution of multiplication on addition

For any three real numbers $a$, $b$, $c$.

$a \times (b + c) = (a \times b) + (a \times c) = a \times b + a \times c$

$(a + b) \times c = (a \times c) + (b \times c) = a \times c + b \times c$
Examples

1. Simplify the following to the simplest form:
   A. \(2\sqrt{5} \ (3 + \sqrt{5})\)
   B. \((\sqrt{2} + 5) \ (3 + \sqrt{2})\)
   C. \((2 - 3\sqrt{5})^2\)

Solution

A. \(2\sqrt{5} \ (3 + \sqrt{5}) = 2\sqrt{5} \times 3 + 2\sqrt{5} \times \sqrt{5}\)
   \(= 2 \times 3 \times \sqrt{5} + 2 \times 5 = 6\sqrt{5} + 10\)

B. \((\sqrt{2} + 5) \ (3 + \sqrt{2}) = \sqrt{2} \ (3 + \sqrt{2}) + 5 (3 + \sqrt{2})\)
   \(= \sqrt{2} \times 3 + \sqrt{2} \times \sqrt{2} + 5 \times 3 + 5 \times \sqrt{2}\)
   \(= 3\sqrt{2} + 2 + 15 + 5\sqrt{2}\)
   \(= 3\sqrt{2} + 17 + 5\sqrt{2} = 8\sqrt{2} + 17\)

C. \((2 - 3\sqrt{5})^2 = (2)^2 + 2 \times 2 \times (-3\sqrt{5}) + (-3\sqrt{5})^2\)
   \(= 4 - 12\sqrt{5} + 9 \times 5\)
   \(= 49 - 12\sqrt{5}\)

2. Give an estimation to the result of \((3 + \sqrt{5}) \times (1 + \sqrt{8})\), then check your answer using the calculator.

Solution

First: The estimate of \(\sqrt{5}\) is 2 \(\therefore (3 + \sqrt{5})\) the estimate of 3 + 2 = 5

the estimate of \(\sqrt{8}\) is 3 \(\therefore (1 + \sqrt{8})\) the estimate of 1 + 3 = 4

\(\therefore (3 + \sqrt{5}) (1 + \sqrt{8})\) the estimate of 5 \times 4 = 20

Second: when we use the calculator to find \((3 + \sqrt{5}) \times (1 + \sqrt{8})\)

We find that the result is 20.0459

Therefore, the estimate is reasonable.
Exercises (1-7)

1. Choose the correct sentence:
   A. \(2\sqrt{3} + 3\sqrt{3} = \quad (5\sqrt{3} \text{ or } 5\sqrt{3} \text{ or } 6 \sqrt{3} \text{ or } 5\sqrt{3})\)
   B. \(\sqrt{5} + \sqrt{5} = \quad (\sqrt{10} \text{ or } 5 \text{ or } 2 \sqrt{5} \text{ or } 5\sqrt{5})\)
   C. \(5 + 7\sqrt{2} - 4 + \sqrt{2} = \quad (15 \text{ or } 1 + 7\sqrt{2} \text{ or } 1 + 8\sqrt{2} \text{ or } 1 + 6\sqrt{2})\)
   D. \(-2\sqrt{3} \times \sqrt{3} = \quad (-6 \text{ or } -2\sqrt{3} \text{ or } 2\sqrt{3} \text{ or } 6)\)
   E. \(\frac{6}{\sqrt{3}} = \quad (\sqrt{2} \text{ or } 2 \sqrt{3} \text{ or } 6\sqrt{3})\)
   F. \((2\sqrt{5})^2 = \quad (10 \text{ or } 20 \text{ or } 4\sqrt{5} \text{ or } 40)\)

2. Simplify to the simplest form:
   A. \(\sqrt{2} (5 + \sqrt{2})\)
   B. \(\sqrt{7} (\sqrt{7} + 2)\)
   C. \(-\sqrt{3} (-5 - \sqrt{3})\)
   D. \((\sqrt{2} + 1)(\sqrt{2} - 1)\)

3. Write each of the following numbers where the denominator is an integer:
   A. \(\frac{10}{\sqrt{5}}\)
   B. \(\frac{8}{\sqrt{6}}\)
   C. \(\frac{6}{2\sqrt{3}}\)
   D. \(\frac{\sqrt{2} + 3}{\sqrt{2}}\)

4. Simplify to the simplest form:
   A. \(2\sqrt{3} + 5 + \sqrt{3} - 6\)
   B. \(2\sqrt{7} - 3\sqrt{2} + \sqrt{7} + 5\sqrt{7}\)
   C. \((\sqrt{3} + 2)(\sqrt{3} - 1)\)
   D. \(\sqrt{5} (3 - \sqrt{5}) - 2 (1 + \sqrt{5})\)

5. If \(a = \sqrt{3} + 2\), \(b = \sqrt{3} - 2\) Find the value of each of the following:
   A. \(a + b\)
   B. \(a - b\)
   C. \(ab\)

6. If \(X = \sqrt{15} + 2\), \(Y = 4 - \sqrt{25}\) estimate the value:
   A. \(X, Y\)
   B. \(X \times Y\)
   C. \(X + Y\)

Check the reasonableness of each value using your calculator.
Operations on the square roots

Think and Discuss

If \( a, b \) are two non-negative real numbers, then

First: \( \sqrt{a} \times \sqrt{b} = \sqrt{ab} \)

For example: \( \sqrt{2} \times \sqrt{3} = \sqrt{2 \times 3} = \sqrt{6} \)
\( \sqrt{2} \times \sqrt{10} = \sqrt{2 \times 10} = \sqrt{20} \)
\( \sqrt{15} \times \sqrt{5} = \sqrt{15 \times 5} = \sqrt{75} \)
\( \sqrt{a \times b} = \sqrt{a \times \sqrt{b}} \)

For example: \( \sqrt{20} = \sqrt{4 \times 5} = \sqrt{4} \times \sqrt{5} = 2 \sqrt{5} \)
\( \sqrt{75} = \sqrt{25 \times 3} = \sqrt{25} \times \sqrt{3} = 5 \sqrt{3} \)

Second: \( \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} \) where \( b \neq 0 \)

For example: \( \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{\sqrt{9}} = \frac{1}{3} \sqrt{5} \)
\( \sqrt{\frac{16}{3}} = \frac{\sqrt{16}}{\sqrt{3}} = \frac{4}{\sqrt{3}} \times \sqrt{\frac{3}{3}} = \frac{4 \sqrt{3}}{3} \)

Third: \( \frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{a}}{\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}} = \frac{\sqrt{ab}}{b} \neq 0 \)

For example: \( \frac{\sqrt{18}}{\sqrt{2}} = \frac{\sqrt{18}}{\sqrt{2}} = \sqrt{9} = 3 \)
\( \frac{\sqrt{84}}{\sqrt{7}} = \sqrt{12} = \sqrt{4 \times 3} = \sqrt{4} \times \sqrt{3} = 2 \sqrt{3} \)
Examples

1. Simplify to the simplest form $\sqrt{32} - \sqrt{72} + 6\sqrt{\frac{1}{2}}$

Solution

$$\sqrt{32} - \sqrt{72} + 6\sqrt{\frac{1}{2}} = \sqrt{16 \times 2} - \sqrt{36 \times 2} + 6 \times \sqrt{\frac{1}{2}}$$

$$= \sqrt{16} \times \sqrt{2} - \sqrt{36} \times \sqrt{2} + 6 \times \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= 4\sqrt{2} - 6\sqrt{2} + 3\sqrt{2} = \sqrt{2}$$

2. If $x = 2\sqrt{5} - 1$, $y = 2 + \sqrt{5}$ find the value of $x^2 + y^2$

Solution

$$x^2 = (2\sqrt{5} - 1)^2 = (2\sqrt{5})^2 - 4\sqrt{5} + 1$$

$$= 4 \times 5 - 4\sqrt{5} + 1 = 21 - 4\sqrt{5}$$

$$y^2 = (2 + \sqrt{5})^2 = 4 + 4\sqrt{5} + 5 = 9 + 4\sqrt{5}$$

$$x^2 + y^2 = 21 - 4\sqrt{5} + 9 + 4\sqrt{5} = 30$$

Practice

1. Put each of the following in the form of $a\sqrt{b}$ where $a$ and $b$ are integers, $b$ is the least possible value:

   - A: $\sqrt{28}$
   - B: $\sqrt{75}$
   - C: $\sqrt{54}$
   - D: $\sqrt{1000}$
   - E: $\sqrt{2\sqrt{72}}$
   - F: $\frac{1}{3}\sqrt{162}$

2. Simplify to the simplest form:

   - A: $2\sqrt{18} \times 3\sqrt{2}$
   - B: $\sqrt{5} \times 2\sqrt{10}$
   - C: $3\sqrt{7} \times 2\sqrt{28}$
   - D: $\sqrt{50} + \sqrt{8}$
   - E: $\sqrt{20} - \sqrt{45}$
   - F: $\sqrt{27} + 5\sqrt{18} - \sqrt{300}$

3. Find the value of $X + Y$, $X \times Y$ in each of the following cases:

   - A: $x = 3 + \sqrt{5}$, $y = 1 - \sqrt{5}$
   - B: $x = \sqrt{3} - \sqrt{2}$, $y = \sqrt{3} + \sqrt{2}$
   - C: $x = 5 - 3\sqrt{2}$, $y = 5 - 3\sqrt{2}$
The two conjugate numbers

If \(a\) and \(b\) are two positive rational numbers.

Then each of the two numbers \((\sqrt{a} + \sqrt{b}), (\sqrt{a} - \sqrt{b})\) is a conjugate to the other one.

Then, their sum is \(2\sqrt{a}\) twice the first term

And their product is \((\sqrt{a} + \sqrt{b})\cdot(\sqrt{a} - \sqrt{b}) = (\sqrt{a})^2 - (\sqrt{b})^2 = a - b\)

= The square of the first term - The square of the second term

The product of two conjugates is always a rational number.

If we have a real number whose denominator is written in the form \((\sqrt{a} \pm \sqrt{b})\), we should put it in the simplest form by multiplying both the numerator and denominator by the conjugate of the denominator.

**Practice**

**Complete**

A. \(\sqrt{5} + \sqrt{2}\) their conjugate ( .............. ) and their product is ( .............. )

B. \(5 - \sqrt{3}\) their conjugate ( .............. ) and their product is ( .............. )

C. \(2\sqrt{3} + \sqrt{2}\) their conjugate ( .............. ) and their product is ( .............. )

**Examples**

1. Given \(x = \frac{8}{\sqrt{5} - \sqrt{3}}, \quad y = \frac{2 - \sqrt{3}}{2 + \sqrt{3}}\)

Write both of \(X\) and \(Y\) where the denominator is a rational number, then find \(X + Y\)

**Solution**

\[
x = \frac{8}{\sqrt{5} - \sqrt{3}} = \frac{8}{\sqrt{5} - \sqrt{3}} \times \frac{\sqrt{5} + \sqrt{3}}{\sqrt{5} + \sqrt{3}}
\]

\[
= \frac{8(\sqrt{5} + \sqrt{3})}{(\sqrt{5})^2 - (\sqrt{3})^2} = \frac{8(\sqrt{5} + \sqrt{3})}{5 - 3} = 4(\sqrt{5} + \sqrt{3})
\]
\[
y = \frac{2 - \sqrt{3}}{2 + \sqrt{3}} = \frac{2 - \sqrt{3}}{2 + \sqrt{3}} \times \frac{2 - \sqrt{3}}{2 - \sqrt{3}}
\]
\[
= \frac{(2 - \sqrt{3})^2}{4 - 3} = \frac{4 - 4\sqrt{3} + 3}{1} = 7 - 4\sqrt{3}
\]
\[
x + y = 4\sqrt{5} + 4\sqrt{3} + 7 - 4\sqrt{3} = 4\sqrt{5} + 7
\]

Given \( x = \frac{4}{\sqrt{7} - \sqrt{3}} \), \( y = \sqrt{7} - \sqrt{3} \),

prove that \( x \) and \( y \) are conjugates, then find the values of:
\( x^2 - 2xy + y^2 \), \( (x - y)^2 \). What do you observe?

**Solution**

\[
x = \frac{4}{\sqrt{7} - \sqrt{3}} \times \frac{\sqrt{7} + \sqrt{3}}{\sqrt{7} + \sqrt{3}} = \frac{4 (\sqrt{7} + \sqrt{3})}{7 - 3} = \sqrt{7} + \sqrt{3}
\]
\[
y = \sqrt{7} - \sqrt{3} \quad \therefore \quad x, y \text{ (two conjugate numbers)}
\]
\[
x^2 - 2xy + y^2 = (\sqrt{7} + \sqrt{3})^2 - 2 (\sqrt{7} + \sqrt{3}) (\sqrt{7} - \sqrt{3}) + (\sqrt{7} - \sqrt{3})^2
\]
\[
= (7 + 2\sqrt{21} + 3) - 2 (7 - 3) + (7 - 2\sqrt{21} + 3)
\]
\[
= 10 + 2\sqrt{21} - 8 + 10 - 2\sqrt{21}
\]
\[
= 12
\]
\[
(x - y)^2 = [(\sqrt{7} + \sqrt{3}) - (\sqrt{7} - \sqrt{3})]^2
\]
\[
= [\sqrt{7} + \sqrt{3} - \sqrt{7} + \sqrt{3}]^2 = (2\sqrt{3})^2
\]
\[
= 4 \times 3 = 12
\]

**Remark**: \( x^2 - 2xy + y^2 = (x - y)^2 \)

**Practice**

In the previous example, find the value of each of the following:

A. \( (x + y) \)  
B. \( (x - y) \)  
C. \( (x + y) (x - y) \)  
D. \( x^2 - y^2 \)

What do you observe?
Exercises (1-8)

1. **Choose** the correct answer:

   A. \( \sqrt{50} - \sqrt{18} - \sqrt{2} = \) .......... (\(\sqrt{30}\) or \(2\sqrt{2}\) or 2 or \(2\sqrt{2}\))

   B. \( (\sqrt{7} - \sqrt{5}) (\sqrt{7} + \sqrt{5}) = \) .......... (2 or 12 or 2\(\sqrt{7}\) or -2\(\sqrt{5}\))

   C. \( (\sqrt{8} + \sqrt{2})^2 = \) .......... (\(\sqrt{10}\) or 10 or 18 or \(\sqrt{18}\))

   D. The multiplicative inverse of \(\frac{\sqrt{3}}{6}\) is .......... \((-\frac{\sqrt{3}}{6}\) or 6\(\sqrt{3}\) or 2\(\sqrt{3}\) or -2\(\sqrt{3}\))

   E. The next number in the pattern: \(\sqrt{3}, \sqrt{12}, \sqrt{27}, \sqrt{48}\) is .......... (\(\sqrt{50}\) or \(\sqrt{75}\) or \(\sqrt{60}\) or \(\sqrt{90}\))

2. **Complete** the following to get a correct sentence:

   A. If \(X = 3 + \sqrt{2}\) then the conjugate is .......... and the product of the number by its conjugate is ..........

   B. The multiplicative inverse for \((\sqrt{3} + \sqrt{2})\) in its simplest form is ..........

   C. **Think** If \(x^2 = 5\), then \((x + \sqrt{5})^2 = \) ........... or ...........

   D. **Think** If \(\frac{1}{x} = \sqrt{5} - 2\), then the value of \(X\) in its simplest form is ..........

   E. \(3\sqrt{2} + \sqrt{8} - \sqrt{18} = \) ..........

3. Simplify to the simplest form: \(2\sqrt{5} + 6\sqrt{\frac{1}{3}} - \sqrt{12} - 5\sqrt{\frac{1}{5}}\)

4. If \(x = \frac{4}{\sqrt{7} - \sqrt{3}}\), \(y = \frac{4}{\sqrt{7} - \sqrt{3}}\). Find the value of \(x^2 y^2\)

5. If \(A = \sqrt{3} + \sqrt{2}\), \(B = \frac{1}{\sqrt{3} + \sqrt{2}}\). Find the value of \(A^2 - B^2\) in its simplest form.

6. If \(x = \sqrt{5} + \sqrt{2}\), \(y = \sqrt{5} - \sqrt{2}\).

   Find the value of \(\frac{x+y}{xy - 1}\) in its simplest form

7. If \(x = \sqrt{7} + \sqrt{5}, y = \frac{2}{x}\)

   Find the value of \(\frac{x+y}{xy}\) in its simplest form
Unit One
Lesson Nine
Operations on the cube roots

Think and Discuss

You will learn how
To carry operations on the cube roots.

Key terms
Cube root.

For any two real numbers \( a, b \):

1. \( \sqrt[3]{a} \times \sqrt[3]{b} = \sqrt[3]{a \times b} \)

For example:
\[
\sqrt[3]{5} \times \sqrt[3]{2} = \sqrt[3]{10} \\
\sqrt[3]{3} \times \sqrt[3]{-4} = \sqrt[3]{-12}
\]

2. \( \sqrt[3]{a \times b} = \sqrt[3]{a} \times \sqrt[3]{b} \)

For example:
\[
\sqrt[3]{40} = \sqrt[3]{8 \times 5} = \sqrt[3]{8} \times \sqrt[3]{5} = 2 \sqrt[3]{5} \\
\sqrt[3]{-128} = \sqrt[3]{-64 \times 2} = \sqrt[3]{-64} \times \sqrt[3]{2} = -4 \sqrt[3]{2}
\]

3. \( \frac{\sqrt[3]{a}}{\sqrt[3]{b}} = \sqrt[3]{\frac{a}{b}} \) where \( b \neq 0, a, b \in \mathbb{R} \)

For example:
\[
\frac{\sqrt[3]{12}}{\sqrt[3]{3}} = \sqrt[3]{\frac{12}{3}} = \sqrt[3]{4}
\]

4. \( \sqrt[3]{\frac{a}{b}} = \frac{\sqrt[3]{a}}{\sqrt[3]{b}} \) where \( b \neq 0, a, b \in \mathbb{R} \)

For example:
\[
\sqrt[3]{\frac{3}{2}} = \frac{\sqrt[3]{3}}{\sqrt[3]{2}}
\]

Think: If we multiply both the numerator and the denominator by \( \sqrt[3]{4} \), then find the product in its simplest form.
Examples:

Simplify to the simplest form:

A. \( \sqrt{54} + 8 \sqrt{-\frac{1}{4}} + 5 \sqrt{16} \)

Solution

A. \( \sqrt{54} + 8 \sqrt{-\frac{1}{4}} + 5 \sqrt{16} = \sqrt{27} \times 2 + 8 \sqrt{-\frac{1}{4}} \times \frac{2}{2} + 5 \sqrt{8} \times 2 \)

\( = \sqrt{27} \times \sqrt{2} + 8 \times \frac{\sqrt{2}}{8} + 5 \times \sqrt{8} \times \sqrt{2} \)

\( = 3 \sqrt{2} + 8 \times \frac{\sqrt{2}}{2} + 5 \times 2 \times \sqrt{2} \)

\( = 3 \sqrt{2} - 4 \sqrt{2} + 10 \sqrt{2} = 9 \sqrt{2} \)

B. \( \sqrt{24} - 6 \sqrt{\frac{8}{9}} = \sqrt{24} - 6 \sqrt{\frac{125}{9}} = \sqrt{8} \times 3 - 6 \times \frac{\sqrt{125}}{\sqrt{9}} \)

\( = \sqrt{8} \times \sqrt{3} - 6 \times \frac{5\sqrt{3}}{3} = 2 \sqrt{3} - 10 = -8 \sqrt{3} \)

Exercises (1-9)

1. Put each of the following in the form \( \sqrt{\frac{a}{b}} \) where \( a \) and \( b \) are two integers, \( b \) is the least positive possible value:

   A. \( \sqrt{54} \)  
   B. \( \sqrt{-1000} \)  
   C. \( \sqrt{128} \)  
   D. \( -\sqrt{2160} \)  
   E. \( \sqrt{1715} \)  
   F. \( \sqrt{686} \)

2. Find the result for each of the following in the simplest form:

   A. \( \sqrt{125} - \sqrt{24} \)  
   B. \( \sqrt{250} - \sqrt{128} \)  
   C. \( \sqrt{\frac{2}{5}} \times \sqrt{\frac{4}{25}} \)  
   D. \( \sqrt{\frac{3}{4}} + \sqrt{\frac{2}{9}} \)  
   E. \( \frac{1}{2} \sqrt{56} - \sqrt{\frac{7}{27}} \)  
   F. \( \frac{1}{2} \sqrt{10} \times 6 \sqrt{100} \)

3. If \( a = \sqrt{5} + 1 \), \( b = \sqrt{5} - 1 \) find the value of each of the following:

   A. \( (a - b)^8 \)  
   B. \( (a + b)^3 \)

4. Prove that:

   A. \( \sqrt{128} + \sqrt{16} - 2 \sqrt{54} = 0 \)
   B. \( \sqrt{54} \times \sqrt{16} = (\sqrt{4} \times 6) = 1 \)
You will learn how

- To solve applications on square and cube roots.

**Key Terms**

- Circle
- Cuboid
- Cube
- Right circular cylinder
- Sphere

**Think and Discuss**

**The Circle:**

- **Circumference of a circle =** \( 2\pi r \) length unit
- **Area of a circle =** \( \pi r^2 \) square unit

where \( r \) is the length of the radius in a circle, \( \pi \) is the (approximate ratio).

**Examples**

1. **Find** the circumference of a circle whose area is 38.5 cm\(^2\) \((\pi = \frac{22}{7})\)

**Solution**

The area of the circle = \( \pi r^2 \)

\[
38.5 = \frac{22}{7} r^2 \quad \Rightarrow \quad r^2 = \frac{38.5 \times 7}{22} = \frac{49}{4}
\]

\[
\therefore \quad r = \sqrt{\frac{49}{4}} = \frac{7}{2} = 3.5 \text{ cm}
\]

2. In the opposite figure, the circle \( M \) is inside the square \( ABCD \). If the area of the yellow sector is \( 10 \frac{5}{7} \text{ cm}^2 \), find the perimeter of the sector \((\pi = \frac{22}{7})\)

**Solution**

We suppose that the length of the radius in a Circle = \( r \)

\[
\therefore \quad \text{The side length of the square} = 2r
\]
The area of the yellow color = the area of the rectangle AEFD - the area of semi circle

\[ \frac{10}{7} = r \times 2r - \frac{1}{2} \times \frac{22}{7} r^2 \]
\[ \frac{75}{7} = 2 r^2 - \frac{11}{7} r^2 = \frac{3}{7} r^2 \]
\[ \therefore r^2 = 25 \quad \therefore r = 5 \text{ cm} \]

The perimeter of the yellow sectors = \( (AE + AD + DF) + \frac{1}{2} \) the circumference of the circle

\[ = (5 + 10 + 5) + \frac{1}{2} \times 2 \times \frac{22}{7} \times 5 = 35 \frac{5}{7} \text{ cm} \]

**Practice**

1. A circle whose area is \( 64 \pi \text{ cm}^2 \). Find the length of its radius, then find its circumference approximating it to the nearest integer (\( \pi = 3.14 \)).

2. In the figure opposite: \( \overline{AB} \) is the diameter of a semi circle. If the area of that region is \( 12.32 \text{cm}^2 \). Find the circumference of that figure.

3. In the opposite figure: there are two circles have the same center "concentric" of center M. If the lengths of their radii are 3cm and 5cm. Find the area and the circumference of the colored region in the terms of \( \pi \).

**The cuboid**

It is a body whose six faces are of a rectangular shape such that every two opposite faces are congruent.

If the lengths of its edges were \( x, y, z \), then:

- The lateral area = the perimeter of the base \( \times \) the height

**The lateral area =** \( 2 ( x + y ) \times z \) square unit

**The total area =** \( 2 ( xy + yz + Xz ) \) square unit

**The volume of the cuboid =** the area of the base \( \times \) the height

**The volume of the cuboid =** \( x \times y \times z \) cubic unit
A special case: the cube
It is a cuboid whose edges are equal in length. If the length of one edge = L length unit, then:
The area of each face = \( L^2 \) square unit

The lateral area of each face = \( 4L^2 \) square unit

The total area = \( 6L^2 \) square unit, the volume of the cube = \( L^3 \) cubic unit

Examples

Find the total area of a cube whose volume is 125 cm\(^3\)

Solution

The volume of the cube = \( L^3 \) \( \therefore 125 = L^3 \) \( \therefore L = \sqrt[3]{125} = 5 \text{ cm} \)
The total area = \( 6L^2 = 6 \times (5)^2 = 150 \text{ cm}^2 \)

Practice

1. Find the total area of a cuboid whose volume is 720 cm\(^3\) and height 5 cm with a squared shape base.
2. Which is more in volume: A cube of 294 cm\(^2\) area or a cuboid with the following dimensions: \( 7 \sqrt{2}, 5 \sqrt{2}, 5 \text{ cm} \).
3. A rectangular hard piece of paper has a length of 25 cm and a width of 15 cm. A square whose side = 4 cm was cut from each of its four corners. Then, the projected parts were folded to form a shape of a cuboid. Find the volume and the total area of that cuboid.

The right circular cylinder:
It is a body that has two parallel congruent bases each is a circular shaped surface, while its lateral surface is a curved surface called cylindrical surface.

- If \( M, M' \) are the bases of the cylinder, then \( M M' \) is the height of cylinder.
**Let's think**

If \( A \in \) the circle \( M \), \( B \in \) the circle \( M' \),

\[
\frac{AB}{A'B'} \parallel \frac{MM'}{M'M'}
\]

Then, if we cut the lateral cylindrical surface at \( AB \) and we stretch that surface, we get the surface of the rectangle \( AB' \ B' \ A' \).

Then, \( AB = \) height of cylinder, \( A A' = \) the perimeter of the base of the cylinder.

The area of the rectangle \( AB' \ B' \ A' = \) the lateral area of the cylinder

The lateral area of the cylinder = the perimeter of the base \( \times \) height = \( 2\pi r h \) (square unit)

the total area of the cylinder = area of lateral surface + sum of the areas of the two bases

\[
= 2\pi r h + 2\pi r^2
\]

(square unit)

the volume of the cylinder = base area \( \times \) height = \( \pi r^2 h \) (cubic unit)

**Example**

A piece of paper has shape of a rectangle ABCD in which \( AB = 10\text{cm} \), \( BC = 44\text{cm} \). It was folded to form a right circular cylinder such that \( AB \) is congruent to \( DC \). Find the volume of the resulted cylinder. \((\pi = \frac{22}{7})\).

**Solution**

The perimeter of the cylinder base = 44 cm.

\[
2\pi r = 44
\]

\[
2 \times \frac{22}{7} r = 44
\]

\[
\therefore r = 7\text{cm}
\]

The volume of the cylinder = \( \pi r^2 h \)

\[
= \frac{22}{7} \times (7)^2 \times 10
\]

\[
= 1540 \text{ cm}^3
\]

**Practice**

1 Find the volume and the total area of a right circular cylinder in which the length of base radius = 14 cm and the height is 20 cm.
Find the total area of a right circular cylinder of volume 7536 cm³ and height 24 cm ($\pi = 3.14$)

Which is more in volume: a right circular cylinder of radius 7 cm and height 10 cm or a cube whose edge length is equal to 11 cm

The sphere:

It is a body of curved surface in which the points have the same distance (r) from a constant point inside it (the center of the sphere).

If the sphere is cut by a plane passing by its center, then the resulted section is a circle whose center is the center of a sphere where its radius is the radius of a sphere (r).

Volume of the sphere = $\frac{4}{3} \pi r^3$ cubic units.

area of the sphere = $4 \pi r^2$ square units.

Examples

The volume of the sphere is $562.5 \pi$ cm³. Find its surface area.

Solution

the volume of sphere = $\frac{4}{3} \pi r^3$

$562.5 \pi = \frac{4}{3} \times \pi r^3$

$r^3 = \frac{562.5 \times \frac{3}{4}}{421.875}$

$r = \sqrt[3]{421.875} = 7.5$ cm

the surface area of sphere = $4 \pi r^2 = 4 \times \pi (7.5)^2 = 225 \pi$ cm²

Practice

Find the volume and the surface area of a sphere whose diameter is 4.2 cm ($\pi = \frac{22}{7}$)
Exercises (1-10)

1. Choose the correct answer

A. The lateral area of a right circular cylinder whose base diameter length b and height h is .... (\pi b^2h, \pi bh, 2\pi bh, \pi bh^2).

B. The volume of a sphere whose diameter length is 6cm = ..... cm³ (288 , 12 \pi , 36 \pi , 288 \pi)

C. The edge length of a cube whose volume is \(2 \sqrt{2}\) cm³ = ..... cm. (\sqrt{2} , 2, 8, 1.5)

D. The radius length of a right circular cylinder whose volume is 40 \pi cm³ and height 10cm = ..... cm (5 , 3 , 2 , 1).

E. The volume of a cuboid whose dimensions are \(\sqrt{2}, \sqrt{3}, \sqrt{6}\) cm is .......(6 , 36 , 6\sqrt{6} , 18\sqrt{2}).

2. Complete

A. The radius length of a sphere whose volume is \(\frac{9}{2}\pi\) cm³ is ....... cm.

B. A right circular cylinder whose base radius length r and height h, then its lateral area is ..... and its volume is ..... 

C. The total area of a cube of edge length = 4 cm is ..... cm².

D. The lateral area of the cuboid = ..... 

3. A sphere with volume 36 \pi cm³ is placed inside a cube, if the sphere touches the cube’s six faces. Find

A. the radius of the sphere. 

B. the volume of the cube.

4. A metal sphere with diameter 6 cm has got melt and changed into a right circular cylinder with base radius length 3 cm. Find its height.

5. Find the height of a right circular cylinder whose height is equal to its base radius and its volume is 72 \pi cm³.

6. A hollow metal sphere with internal radius 2.1 cm and external radius 3.5 cm. Find its mass to the nearest gram given that the mass of a cubic centimeter of such a metal is 20 gm (\pi \frac{22}{7})
Think and Discuss

First: Solving Equations of first degree in one variable in R

We know that: The equation $3x - 2 = 4$ is called an equation of first degree where the exponent of the (unknown) variable $X$ is 1. To solve that equation in R:

$3x - 2 = 4$  \hspace{1cm} \text{By adding 2 to the sides of the equation}

$3x = 6$  \hspace{1cm} \text{(we can multiply by the multiplicative inverse of the coefficient of X)}

$\frac{1}{3} \times 3x = \frac{1}{3} \times 6$

$\therefore x = 2$

i.e the solution set $\{2\}$

This solution can be graphed on the number line as shown in the figure opposite.

Examples

1. Find the solution set of the equation $\sqrt{3} x - 1 = 2$, in R, then graph the solution on the number line.

Solution

$\sqrt{3} x - 1 = 2$  \hspace{1cm} \therefore $\sqrt{3} x = 3$

$\therefore x = \frac{3}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$  \hspace{1cm} \therefore x = \sqrt{3} \in R$

The solution set is $\{\sqrt{3}\}$

This solution can be graphed on the number line as shown in the figure opposite.
2. **Find** The solution set for the equation \( x + \sqrt{2} = 1 \), in \( \mathbb{R} \), then graph the solution on the number line.

**Solution**

\[
x + \sqrt{2} = 1 \\
\therefore x = 1 - \sqrt{2} \in \mathbb{R}
\]

This can be graphed on the number line as shown in the figure opposite.

**Practice**

1. **Find** the solution set for the following equations in \( \mathbb{R} \), then graph the solution on the line number.

   - A. \( 5x + 6 = 1 \)
   - B. \( 2x + 4 = 3 \)
   - C. \( 2x - 3 = 4 \)
   - D. \( x + 5 = 0 \)
   - E. \( \sqrt{2}x - 1 = 1 \)
   - F. \( x - 1 = \sqrt{5} \)

Second: Solving inequalities of the first degree in one variable in \( \mathbb{R} \), graphing the solution on the number line.

The following properties are used to solve the inequality in \( \mathbb{R} \). The solution set is written in the form of an interval.

**If** \( A, B, C \) **were real number where** \( A < B \), then:

1. \( A + C < B + C \). **addition property.**
2. If \( C > 0 \) then \( A \times C < B \times C \). **property of multiplication by a positive real number.**
3. If \( C < 0 \) then \( A \times C > B \times C \). **property of multiplication by a negative real number.**

**Examples**

1. **Find** the solution set for the inequality \( 2x - 1 \geq 5 \) in \( \mathbb{R} \) and represent the solution set graphically.

   **Solution**

   By adding 1 to the sides of the inequality it becomes \( 2x \geq 6 \) by multiplying the side of the inequality by \( \left( \frac{1}{2} > 0 \right) \) \( x \geq 3 \)

   \( \therefore \) The solution set in \( \mathbb{R} \) is \([3, \infty[\)

   and it is graphed by green color ray on the number line.
2. **Find** the solution set for the inequality \( 5 - 3x > 11 \), in \( \mathbb{R} \), then represent the solution graphically.

**Solution**

By adding \((-5)\) to the sides of the inequality then \(-3x > 6\)

by multiplying the sides of the inequality by \(-\frac{1}{3} < 0\) we get:

\[ \therefore x < -2 \]

i.e., the solution set in \( \mathbb{R} \) is \(-\infty, -2\]

and it is represented by the green color ray on the number line.

3. **Find** the solution set for the inequality \(-3 \leq 2x - 1 < 5\) in \( \mathbb{R} \) and represent the solution graphically.

**Solution**

by adding \((1)\) to the sides of the inequality -3 + 1 \(\leq\) 2x - 1 + 1 \(\leq\) 5 + 1

Namely, \(-2 \leq 2x < 6\), and by multiplying the sides of the inequality by \(\frac{1}{2} > 0\) \(-1 \leq x < 3\)

\[ \therefore \text{the solution set in } \mathbb{R} \text{ is } [-1, 3]\text{ and it is graphed on the number line by the green color.} \]

**Example**

What is the solution set for the inequality in \( \mathbb{N} \)?

What is the solution set for the inequality in \( \mathbb{Z} \)?

**Exercises (1-11)**

1. **Complete** the following to have a true sentence where \( X \in \mathbb{R} \)

   A. If \( 5x < 15 \), then \( x \) ............
   
   B. If \( x - 3 \geq 4 \), then \( x \) ............
   
   C. If \( -2x < 3 \), then \( x \) ............
   
   D. If \( 1 - x > 4 \), then \( x \) ............
   
   E. If \( \sqrt{2} x < 4 \), then \( x \) ............
Unit 1: Lesson 11

Find The solution set for each of the following inequalities in \( \mathbb{R} \) in the form of intervals, then graph the solution on the number line:

A \( 3x - 1 < 5 \)  
B \( 2x + 5 > 3 \)
C \( 2x + 3 < 1 \)  
D \( 5 - x > 3 \)
E \( 1 - 5x < 6 \)  
F \( \frac{1}{2} x + 1 < 2 \)

Find the solution set for each of the following inequalities in \( \mathbb{R} \) in the form of an interval, then represent the solution on the number line:

A \( -1 < 2x + 1 < 5 \)  
B \( -5 < 2x - 3 < 1 \)
C \( -3 < 4x - 7 < 5 \)  
D \( 4 < 3x + 4 < 7 \)
E \( 1 < 5 - 2x < 3 \)  
F \( 1 < 3 - 2x < 5 \)

Find the solution set for each of the following inequalities in \( \mathbb{R} \), write it in the form of an interval, then represent the solution on the number line:

A \( -3 \leq x < 3 \)  
C \( \sqrt[3]{-8} \leq x + 1 \leq \sqrt{9} \)
B \( -3 \leq x < 3 \)  
D \( 5 < 3 - x \leq 3^2 \)

General Exercises

Complete the following to have a true sentence:

A \( \sqrt{9} + \sqrt[3]{-8} = \) ............
B A vessel that has the shape of a cub whose capacity is 8 liters, the length of the edge of its interior face is ............ cm.
C The solution set for the equation \( x^2 + 9 = 0 \) in \( \mathbb{R} \) is ............
D \( (\sqrt{3} + \sqrt{2})^2 + (\sqrt{3} - \sqrt{2})^2 = \) ............
E A rectangle of dimensions are \( (\sqrt{5} + 1), (\sqrt{5} - 1) \) cm has an area of ............ cm².
F \( \sqrt{54} - \sqrt{-16} = \) ............
G \( [-1, 5] - [-1, 5] = \) ............
H The solution set for the equation \( \sqrt{2} x - 1 = 3 \) in \( \mathbb{R} \) is ............
I A sphere whose diameter is equal to 6 L (length unit) has a volume of ............ (cubic unit).
J \( |\sqrt{-125}| = \sqrt{\)
2. **Find** the solution set for each of the following inequalities in \( \mathbb{R} \), in the form of an interval, then represent the solution on the number line:

- **A** \( 5x - 3 < 2x + 9 \)
- **B** \( 3 - 4x \leq x - 2 \)
- **C** \( x \leq 2x - 1 \leq x + 3 \)
- **D** \( x - 1 < 3x - 1 \leq x + 1 \)
- **E** \( 4x \leq 5x + 2 < 4x + 3 \)
- **F** \( 5x + 7 > 6x > 5x \)

3. If \( x = \frac{\sqrt{6} + \sqrt{5}}{\sqrt{6} - \sqrt{5}} \), prove that \( x + \frac{1}{x} = 22 \)

4. **Find** in the simplest form: \( \sqrt{54} + 4\sqrt{\frac{1}{4}} - \sqrt{2} \)

5. **Find** the total area of a right circular cylinder of volume \( 72 \pi \) cm\(^3\), and height 8 cm.

6. **Find** the following using the number line \([3, 6] \cap [4, 7]\):

7. If \( x = \frac{5\sqrt{2} + 3\sqrt{5}}{\sqrt{5}} \), \( y = \frac{2\sqrt{5} - 3\sqrt{2}}{\sqrt{2}} \) Find the value
   - **A** \( x^2 + y^2 \)
   - **B** \( x \cdot y \)
   and prove that \( x^2 + y^2 = 38 \times y \)

8. If \( X = \sqrt{5} + 2 \), \( y = \sqrt{5} - 2 \) Find the value of \( (x + y)^3 + (x - y)^3 \).

9. If \( x = \sqrt{5} - \sqrt{3} \), \( y = \frac{2}{\sqrt{5} - \sqrt{3}} \), Find the value of \( (x^2 + 2xy + y^2) \)

10. If \( A = \sqrt{3} + \sqrt{2} \), \( B = \sqrt{3} - \sqrt{2} \),
    find the value of \( (A^2 - AB + B^2) \)

11. If \( x = \frac{3\sqrt{5} + 5\sqrt{2}}{\sqrt{5}} \), \( y = \frac{2\sqrt{5} - 3\sqrt{2}}{\sqrt{2}} \)
    Prove that \( \frac{x^2 + y^2}{xy} = 38 \)
Find: \( \sqrt{27} + \sqrt{12} + \sqrt{0.125} \)

Open Microsoft Excel and record the shown numbers in cells A1, D1 and B1.

To find the cube root of cell A1, write the formula \( A1^{(1/3)} \) in the cell F1, then Enter... the result is 3.

To find the square root of cell B1, write the formula \( B1^{(1/2)} \) in the cell H2, then Enter... the result is 3.5.

To find the cube root of cell J1, write the formula \( D1^{(1/3)} \) in the cell J2, then Enter... the result is 0.5.

Write down the sum of \( F2+H2+J2 \) in cell L2, after you do a click on = then the sum is 1.

Activity

Activity Draw a regular hexagon ABCDEF whose side is equal to 4 cm.

1. Find the measure of its interior angle.
2. Draw its diagonals \( \overline{AD} \), \( \overline{BE} \), \( \overline{CF} \) then deduce the length of each diagonal without measuring them by the ruler.
3. Draw a circle that passes by its vertices.
4. Find its area.
Unit Test

1. Complete to get a true sentence:
   A. \( [ -3, 2] \cap \mathbb{R}^* = \ldots \ldots \ldots \ldots \)
   B. The multiplicative inverse for \( \frac{-\sqrt{2}}{6} \) is \( \ldots \ldots \ldots \ldots \)
   C. \( \sqrt{5}, \sqrt{20}, \sqrt{45}, \sqrt{80}, \ldots \ldots \ldots \ldots \) Complete in the same pattern.
   D. If \( x = \sqrt{3} + 7, y = \sqrt{3} - 7 \) then \( (x + y)^3 = \ldots \ldots \ldots \ldots \)
   E. The circle of 20X cm circumference has an area of \( \ldots \ldots \pi \) cm²

2. Choose the correct sentence from Parentheses:
   A. A cube of 64 cm³ has a lateral area of \( \ldots \ldots \) cm² (4 or 8 or 64 or 96)
   B. \( \sqrt{12} - \sqrt{3} = \ldots \ldots \ldots \ldots \) \( \left( 3 \text{ or } \sqrt{3} \text{ or } 2\sqrt{3} \text{ or } 3\sqrt{3} \right) \)
   C. The multiplicative inverse for \( \frac{-\sqrt{6}}{12} \) is \( \ldots \ldots \ldots \ldots \) \( \left( -\frac{12}{\sqrt{6}} \text{ or } \frac{\sqrt{6}}{12} \text{ or } -2\sqrt{6} \text{ or } 2\sqrt{6} \right) \)
   D. \( \sqrt{54} + \sqrt{2} = \ldots \ldots \ldots \ldots \) \( \left( \sqrt{52} \text{ or } \sqrt{2} \text{ or } 2\sqrt{2} \text{ or } 4\sqrt{2} \right) \)
   E. \( [-3, 4] - [-3, 5] = \ldots \ldots \ldots \ldots \) \( \left( -3, 4 \text{ or } -3, 4 \right) \) or \( -3, 5 \text{ or } -3, 4 \)

3. Reduce to the simplest from \( 2\sqrt{18} + \sqrt{50} + \frac{1}{3} \sqrt{162} \)

4. A lead cuboid in which the lengths of its faces are 77cm, 24cm and 21cm. It was melted to form a sphere. Find the radius of that sphere. \( (\pi = \frac{22}{7}) \)

5. If \( A = \frac{4}{\sqrt{7} - \sqrt{3}}, B = \frac{4}{\sqrt{7} + \sqrt{3}} \) find the value of \( \frac{A - B}{AB} \)

6. Using the number line find \( -1, 3 \cup [0, 5] \) in the form of an interval

7. Find the lateral area for a right circular cylinder of volume 924cm³ and height 6cm \( (\pi = \frac{22}{7}) \).

8. If \( x = \sqrt{10} + 2, y = \sqrt{26} - 1 \) Give an estimate for the product of \( X \times Y \) and use the calculator to find the difference between your estimate and the exact answer.

9. Find the solution set in \( R \) and graph the solution on the number line.
   A. \( 1 < 2x + 3 \leq 9 \)
   B. \( x + 2\sqrt{3} = 3 \)
UNIT TWO

Relation Between Two Variables
Unit TWO
Lesson One

Linear Relation of two variables

Think and Discuss

You will learn how
- The linear relations of two variables
- To graph the linear relations of two variables

key terms
- Variable
- Relation
- Linear equation

A person has some bills of LE 50 and LE 20. He bought an electrical apparatus for LE 390.

Think: How many bills of each type does he give to the seller?

Suppose: \( x \) represents the number of fifties bills, then the value of what he has of these bills is LE 50x, \( y \) represents the number of Twenties bills, then the value of what he has of these bills is LE 20y.

Required is to know: \( x \) and \( y \) that verify the equation:

\[
50x + 20y = 390
\]

This relation represents a linear equation in two variables. Dividing both sides over 10 produces the following equivalent equation:

\[
5x + 2y = 39
\]

\[
\therefore y = \frac{39 - 5x}{2}
\]

Note that: \( x \) and \( y \) are natural numbers. Therefore, \( x \) should be an odd number.

The following table can be created to know the different possibilities of giving bills to the seller: a bill of LE 50 and 17 bills of LE 20, or 3 bills of LE 50 and 12 bills of LE 20, or 5 bills of LE 50 and 7 bills of LE 20, or 7 bills of 50 and 2 bills of LE 20.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>((x, y))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>17</td>
<td>(1, 17)</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
<td>(3, 12)</td>
</tr>
<tr>
<td>5</td>
<td>7</td>
<td>(5, 7)</td>
</tr>
<tr>
<td>7</td>
<td>2</td>
<td>(7, 2)</td>
</tr>
<tr>
<td>9</td>
<td>negative</td>
<td>refused</td>
</tr>
</tbody>
</table>
Practice

1. A person has some bills of L.E 5 and some of L.E20. He bought some goods from a shopping center for L.E75. What are the different possibilities of paying this amount in the two types of bills which he has?

2. The perimeter of an isosceles triangle is 19cm. What are the different possible lengths of its sides? Side length $\in \mathbb{Z}_+$

Remember: The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

The Relation of two variables

$a \times b^y = c$ where $a \neq 0, b \neq 0$

is called a linear relation of two variable $x$ and $y$ and can be described by a set of ordered pairs $(x, y)$ verifying this relation.

Example:

Refer to the relation $2x - y = 1$

- If $x = 1$, then $y = 1$.
  - $(1, 1)$ satisfies the relation.
- If $x = 0$, then $y = -1$.
  - $(0, -1)$ satisfies the relation.
- If $x = 3$, then $y = 5$.
  - $(3, 5)$ satisfies the relation.
- If $x = -1$, then $y = -3$.
  - $(-1, -3)$ satisfies the relation.

Thus, there are an infinite number of ordered pairs satisfying the relation.

Note that:

- The linear relation $2x - y = 1$, can be represented graphically by using any of the ordered pairs obtained before.
- Each point in the straight line (in red) is represented by an ordered pair whose elements satisfy the linear relation $2x - y = 1$. 
Practice

1. Find four ordered pairs satisfy each linear relation and represent it graphically:
   - A. \( x + y = 3 \)
   - B. \( x - 2y = 5 \)
   - C. \( y = 2 \)
   - D. \( x = 1 \)

2. Find the value of \( b \), where \((-3, 2)\) satisfies the relation \( 3x + by = 1 \).

3. Find the value of \( k \), where \((k, 2k)\) satisfies the relation \( x + y = 15 \).

Graphing the Relation of two Variables

The relation \( ax + by = c \) where \( a \) and \( b \) or both are not equal zero.
is called a linear relation of two variables \( x \) and \( y \) and can be represented graphically

by a straight line.

- for \( a = 0 \)
  - The relation is represented by a straight line parallel to \( x \)-axis.
  - Example: \( 2y = 3 \)
    - i.e.: \( y = \frac{3}{2} \)
    - is represented by the red line which passes through the point \((0, \frac{3}{2})\) and is parallel to \( x \)-axis.

- Special case:
  - the relation \( y = 0 \) represents the \( x \)-axis

- for \( b = 0 \)
  - The relation is represented by a straight line parallel to \( y \)-axis.
  - Example: \( x = -2 \)
    - is represented by the red line which passes through the point \((-2, 0)\) and is parallel to \( y \)-axis.

- Special case:
  - the relation \( x = 0 \) represents the \( y \)-axis.

Practice

1. Graph each relation of the following:
   - A. \( 2x = 5 \)
   - B. \( y + 1 = 0 \)
Find the relation that is represented by the red line in each figure below:

Example:

Graph the relation: \( x + 2y = 3 \)

Solution

Choose some ordered pairs that satisfy the relation:

For \( y = 2 \) \( \therefore x = -1 \) \((-1, 2)\)

\( y = 0 \) \( \therefore x = 3 \) \((3, 0)\)

\( y = -1 \) \( \therefore x = 5 \) \((5, -1)\)

The following table lists these data:

<table>
<thead>
<tr>
<th>( x )</th>
<th>-1</th>
<th>3</th>
<th>5</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>2</td>
<td>0</td>
<td>-1</td>
<td>( \frac{3}{2} )</td>
</tr>
</tbody>
</table>

The red line represents this relation.

Discuss with your teacher:

1. What happens to the value of \( y \) when increasing the value of \( x \)?
2. When does the line representing the relation \( ax + by = c \) pass through the origin 0?

Exercises (2 - 1)

1. Graph each of the following:
   - A. \( x + y = 2 \)
   - B. \( 2x - y = 3 \)

2. Graph the relation \( 2x + 3y = 6 \). If the straight line representing this relation intersects the x-axis in point A and the y-axis in point B. Find the area of the triangle OAB, where 0 is the origin.
The Slope of a line and real-life Applications

Think and Discuss

You will learn how:
1. The slope of a line.
2. Real-life applications on the slope of a line.

Key terms:
- Slope.
- Positive slope.
- Negative slope.
- Zero-slope.
- Undefined slope.

When observing the motion of a point on a straight line from the location A \((x_1, y_1)\) to the location B \((x_2, y_2)\), where \(x_2 > x_1\) and A, B ∈ line, then:

1. the change in x-coordinate = \(x_2 - x_1\), and is called the horizontal change.
2. the change in y-coordinate = \(y_2 - y_1\), is called the vertical change and may be positive, negative or zero.

The slope of a line:

\[
S = \frac{\text{change in y-coordinate}}{\text{change in x-coordinate}} = \frac{y_2 - y_1}{x_2 - x_1}
\]

where \(x_2 \neq x_1\)

In the following examples you will learn different cases of the vertical change \((y_2 - y_1)\):

Example (1):
If: A \((-1, 1)\) and B \((2, 3)\), then: the slope of \(AB\)

\[
S = \frac{3 - 1}{2 - (-1)} = \frac{2}{3}
\]
Note that:
1. The point $A$ moves on the line upwards to the point $B$.
2. $y_2 > y_1$  
3. The slope of the line is positive.

**Example (2):**

If: $A (0, 2), B (2, 1)$;  
then: the slope of $\frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 2}{2 - 0} = \frac{0}{2} = 0$

**Not that:**
The point $A$ moves on the line downwards to the point $B$.  
2. $y_2 < y_1$  
3. The slope of the line is negative.

**Example (3):**

If: $A (-1, 2)$ and $B (3, 2)$,  
then: the slope of the line $\frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 2}{3 - (-1)} = \frac{0}{4} = 0$

**Not that:**
1. The point $A$ moves horizontally to point $B$.  
2. $y_2 = y_1$  
3. The slope of the line = zero

**Example (4):**

If: $A = (2, 1)$ and $B(2, 3)$ then: we cannot calculate the slope. Because the definition of the slope is conditioned to have a change in the x-coordinate i.e. $x_2 - x_1 \neq 0$

**Not that:**
1. The point $A$ moves vertically to point $B$.  
2. $x_2 = x_1$  
3. The slope of the line is an undefined number.
**Practice:**

1. Find the slope of the straight line $AB$ in each of the following cases:
   - **A** (1, 2), B (5, 0).
   - **B** A (2, -1), B (4, -1).
   - **C** A (-1, 3), B (2, 1).
   - **D** A (3, -1), B (3, 2).

2. Find the slope of $AB$, $BC$, and $AC$, where A (2, -1), B (3, 2), and C (4, 5) and represent each line graphically. What do you observe?

3. Choose the true answer:

   **First:** The following table shows the relation between $x$ and $y$ as follows:
   \[
y = x + 4 \text{ or } y = x + 1 \text{ or } y = 2x - 1 \text{ or } y = 3x - 2
   \]
   
   **Second:** If (2, -5) satisfies the relation $3x - y + c = 0$, then $c = (1, 11, -11)$.

   **Third:** (3, 2) does not satisfy the relation \((y + x = 5, \ 3y - x = 3, y + x = 7, \ y - x = 1)\)

   **Fourth:** An irrigation machine consumes 2.47 litres of diesel to work for 3 hours. If the machine works for 10 hours, it consumes... litres. (7.2, 8, 8.4, 9.6)

4. Find the slope of the line $AB$, where A(-1, 3) and B (2, 5).

   Is the point $c (8, 1) \in AB$?

**Real-life Applications on the slope of a line.**

**Application (1):**

The opposite figure shows capital change of a company during 8 years.

A. Find the slope of $AB$, $BC$ and $CD$.

   What is the meaning of each?

B. Find the starting capital of the company.

**Solution**

A (0, 20), B (4, 60), C (6, 60), D (8, 50)
Unit 2 : Lesson 2

First: The slope of $\frac{AB}{4} = \frac{60 - 20}{4} = 10$, shows the increasing of the capital during the first four years with a rate of 10 thousand pound.

The slope of $\frac{BC}{6} = \frac{60 - 60}{6 - 4} = 0$, means that the capital was constant during the fifth and sixth years.

The slope of $\frac{CD}{8} = \frac{50 - 60}{8 - 6} = -5$, shows the decreasing of the capital during the last two years with a rate of 5 Thousand pound.

Second: Starting Capital = the y-coordinate of the point A = LE 20,000

Practice:

The opposite figure shows the relation between the height of a person (in cm) and his age (in years).

First: Find the slope of $\frac{AB}{2}$, $\frac{BC}{6}$ and $\frac{CD}{8}$

What is the meaning of each?

Second: Calculate the difference between the height of this person as he was 8 years old and his height as he was 30 years old.

Application (2):

Hazem filled up the 40 Litres tank of his car. As covering a distance of 120 km, the fuel gage shows the rest of fuel is $\frac{3}{4}$ of the tank. Draw a diagram to show the relation between the amount of fuel in the tank and covered distance (This relation is linear). Calculate the covered distance as the tank is totally getting empty.

Solution

On the starting point: $A (0, 40)$

After covering 120 km $B = (120, 30)$

The slope of $\frac{AB}{120} = \frac{30 - 40}{120 - 0} = -\frac{1}{12}$

This slope means the fuel amount decreases with a rate of 1L per 12 km, which means 1L is enough to cover a distance of 12 km.
The covered distance that makes the tank empty is calculated as follows:

\[
\text{Fuel Amount} \div \text{Decreasing Rate} = \frac{40}{\frac{1}{12}}
\]

\[= 40 \times \frac{12}{1} = 480 \text{km.}
\]

Note that: \[\overrightarrow{AB}\] intersects the distance-axis in the point \((480, 0)\) which gives the required distance.

**Exercises (2 – 2)**

1. Complete to make a true statement:
   - A. Let \(A(1, 3)\) and \(B(2, 1)\). Then the slope of \(\overrightarrow{AB}\) =
   - B. If \((-1, 5)\) satisfies the relation \(3x + ky = 7\), then \(k =
   - C. The slope of any line parallel to \(x\)-axis =
   - D. The slope of any line parallel to \(y\)-axis =
   - E. If \(A, B\) and \(C\) are collinear points. Then the slope of \(\overrightarrow{AB}\) = the slope of

2. Essam has 10 bills of LE 5 and other bills of LE 20. He bought some goods from a shopping center for LE 65. Determine the different possibilities to pay this amount of money. Find the relation and graph it.

3. The selling price of a computer table is LE 100 and its chair is LE 50. If the store sells in one week with LE 500. What are the represented expectations to the number of computer tables and chairs?
   Represent the relation graphically.

4. In the opposite figure \(ABC\) is a triangle. Complete by using one of the following words:
   (positive, negative, zero, undefined)
   - A. The slope of \(\overrightarrow{AB}\) =
   - B. The slope of \(\overrightarrow{BC}\) =
   - C. The slope of \(\overrightarrow{AO}\) =
   - D. Slope of \(\overrightarrow{AC}\) =
5. In the opposite figure:
LMN is a right angled triangle at L, where \( m(\angle M) = 45^\circ \), given that \( L (3, 2) \) and \( M (7, 2) \). Find the coordinates of \( N \) and calculate the slope of \( MN \).

6. The following diagrams shows the relation between the covered distance (in m) and the elapsed time (in sec) of an object. Determine the position of the object at the starting motion and its position after 6 seconds when \( t = 6 \) sec. Find the slope of the line in each case, and state what it represents.

Technology:

1. Open Microsoft Office Excel to draw the two axes x-y then write the shown number as in fig(1) in the first column A and column B.

2. Do a mouse click to shade the two columns then from the menu INSERT choose CHART as in fig (2) then XY SCATTER as in fig (3) then press NEXT and FINISH, the x-y axes appear.

3. Do a mouse click on \( \square \) from the menu of drawing downward of the page EXCEL and determine the values of dots as shown in fig (4)

4. Do a mouse click on \( \text{Graph} \)

   A. Draw a straight line passes through \( (2, 1) \) and \( (0, 2) \) then the slope is equal to \( 2 - 1 )/(0 - 2) \) which is equal to \( -\frac{1}{2} \) of the blue line.

   B. Draw a straight line passes through \( (2, 2) \) and \( (-2, 2) \), then the slope is equal to \( (2 - 2)/(2 - 2) \) which is equal to zero. i.e. the line is parallel to the x-axis - the yellow line.

   C. Draw a straight line passes through \( (2, -1) \) and \( (2, 5) \), then the slope is equal to \( (2 - 1)/(5 - 1) \) \( (2 - 2) \), then the slope is undefined, i.e. the line is parallel to the y-axis - the red line.
Activity

The opposite diagram shows the relation between the covered distance (in km) and elapsed time (in h) of two trains A and B over the distance between two train stations. Use the diagram to find:

A. The distance between the two train stations.
B. The elapsed time of each train.
C. The average speed of each train.
D. The meaning of the horizontal segment in the diagram of train A.

The average speed = \frac{\text{The covered distance}}{\text{Total elapsed time needed to cover this distance}}

Unit Test

1. Choose the true answer:
   A. Which of the following ordered pairs satisfies the relation \( 2x + y = 5 \)
      \((-1, 3), (1, 3), (3, 1), (2, 2)\)
   B. Which of the following relations is the relation illustrated in the table.
      \((y = x + 7, \ y = x - 7, \ y = 3x + 1, \ y = x + 1)\)
   C. Let \(A(3, 5)\) and \(B(5, -1)\), then the slope of \(\overrightarrow{AB} = (-\frac{1}{3}, -3, \frac{1}{3})\)
   D. The line represented by the relation \(3x + 8y = 24\), intersects the y-axis at the point \(((0, 8), (8, 0), (0, 3), (3, 0))\)

2. Let \(A(2, -1), B(10, 3)\) and \(C(2, 3)\). Find the slope of \(\overrightarrow{AB}, \overrightarrow{BC}\) and \(\overrightarrow{AC}\).

3. Atef filled up the 50 L-tank of his car. As covered a distance of 100km, the fuel gage shows the rest of fuel is \(\frac{4}{5}\) of the tank. Draw a diagram to show the relation between the amount of fuel remaining in the tank and the distance covered. What is the covered distance as the tank is getting totally empty?
UNIT THREE

3

Statistics
Think and Discuss

If you study the traffic jam problem and its possible solutions:

- What are the sources of your data?
- How can you collect data about such a problem?
- What are the statistical methods you will use to analyze the data?
- Can you explain the results you collected?
- What do you suggest to solve that problem and improve traffic fluidity?

Collecting data

Let's work together Cooperate with your classmates on collecting data from their sources through distribution of roles:

A. **Group 1:** Collects primary data about the problem under discussion through a survey that asks about (the means of transportation - Roads conditions - time of traffic jam - Existence of traffic signs - existence of security).

B. **Group 2:** Collects secondary data about the problem under discussion from the traffic reports - the internet - the mass media).

C. **Group 3:** Observes the crowdest roads, the drivers' behavior and their obedience to traffic rules the pedestrians' commitment to the virtues to the road as well as crossing the roads at safe places.
Cooperate with your classmates on making a frequency table that represents the means of transportation used by your classmates.

<table>
<thead>
<tr>
<th>Means of transportation</th>
<th>Subway</th>
<th>bus</th>
<th>Private Car</th>
<th>Taxi</th>
<th>bicycle</th>
<th>on foot</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>........</td>
<td>........</td>
<td>........</td>
<td>........</td>
<td>........</td>
<td>........</td>
<td>........</td>
</tr>
</tbody>
</table>

Determine the most used means of transportation (The mode)

1. Is that means suitable? does it help solving the traffic jam problem? why?

2. What do you suggest to solve this problem according to the results you have collected?

Example

Below are the scores of 30 students in an examination

```
7  10  7  4  5  8  6  7  13  12
2  9  11 12 11 9  15 12 13  9
5 14 19  3  9 14  3 13  8 17
```

Required: forming a frequency table with sets that represents that data.

Solution

To form a frequency table with sets, follow the following steps:

First: find the highest and the lowest values of the collected data?

let the previous collected data be \( X \)

then: \( X = \{x : 2 \leq x \leq 19\} \)

i.e: \( X \) values begins with 2 and ends in 19

i.e: the range = the highest value - the lowest value = \( 19 - 2 = 17 \)

Second: divide set \( X \) into a number of separate subsets each of them is equal in range.

let them be 6 sets. \( \Rightarrow \) The range of the set = \( \frac{17}{6} \) i.e approximated to 3
Third: the subsets are as follow.

<table>
<thead>
<tr>
<th>The first set</th>
<th>2 -</th>
<th>The third set</th>
<th>8 -</th>
</tr>
</thead>
<tbody>
<tr>
<td>The second set</td>
<td>5 -</td>
<td>The Fourth set</td>
<td>11 -</td>
</tr>
</tbody>
</table>

Remark: 2- means the set of data greater than or equal to 2 and less than 5 and so on.

Fourth: Record the data in the following table:

<table>
<thead>
<tr>
<th>Set</th>
<th>Tally</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 -</td>
<td>/////</td>
<td>4</td>
</tr>
<tr>
<td>5 -</td>
<td>////</td>
<td>6</td>
</tr>
<tr>
<td>8 -</td>
<td>////</td>
<td>7</td>
</tr>
<tr>
<td>11 -</td>
<td>/////</td>
<td>8</td>
</tr>
<tr>
<td>14 -</td>
<td>///</td>
<td>3</td>
</tr>
<tr>
<td>17 -</td>
<td>///</td>
<td>2</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>30</td>
</tr>
</tbody>
</table>

Fifth: Delete the tally column from the table to get the frequency table with sets. It can be written either vertically or horizontally. The following is the horizontal form of the table:

<table>
<thead>
<tr>
<th>Sets</th>
<th>2 -</th>
<th>5 -</th>
<th>8 -</th>
<th>11 -</th>
<th>14 -</th>
<th>17 -</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>4</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>3</td>
<td>2</td>
<td>30</td>
</tr>
</tbody>
</table>

Exercises (3 - 1)

1. Below are the weekly wages of 40 workers in a factory in L.E

<table>
<thead>
<tr>
<th>47</th>
<th>71</th>
<th>36</th>
<th>94</th>
<th>54</th>
<th>64</th>
<th>87</th>
<th>89</th>
<th>62</th>
<th>57</th>
</tr>
</thead>
<tbody>
<tr>
<td>51</td>
<td>61</td>
<td>44</td>
<td>52</td>
<td>70</td>
<td>66</td>
<td>56</td>
<td>32</td>
<td>69</td>
<td>36</td>
</tr>
<tr>
<td>79</td>
<td>48</td>
<td>77</td>
<td>90</td>
<td>65</td>
<td>99</td>
<td>96</td>
<td>67</td>
<td>60</td>
<td>55</td>
</tr>
<tr>
<td>95</td>
<td>75</td>
<td>81</td>
<td>84</td>
<td>78</td>
<td>38</td>
<td>49</td>
<td>94</td>
<td>48</td>
<td>59</td>
</tr>
</tbody>
</table>

Required: Form a frequency table with sets (use the subsets: 30-, 40-, 50-, .......90-). What is the set with the highest frequency? what is the set with lowest frequency?
2. The following are the scores of 30 students in a monthly math exam.

<table>
<thead>
<tr>
<th>25</th>
<th>35</th>
<th>40</th>
<th>20</th>
<th>30</th>
<th>37</th>
<th>40</th>
<th>33</th>
<th>22</th>
<th>38</th>
</tr>
</thead>
<tbody>
<tr>
<td>35</td>
<td>36</td>
<td>28</td>
<td>37</td>
<td>39</td>
<td>28</td>
<td>32</td>
<td>26</td>
<td>29</td>
<td>37</td>
</tr>
<tr>
<td>23</td>
<td>34</td>
<td>35</td>
<td>36</td>
<td>29</td>
<td>38</td>
<td>40</td>
<td>35</td>
<td>37</td>
<td>31</td>
</tr>
</tbody>
</table>

Required:
A. Form a frequency table with sets for these scores.
B. Find the total number of excellent students. The excellence rate is 36 marks or more.

3. The following table shows the days-off which 40 workers got during a year.

<table>
<thead>
<tr>
<th>15</th>
<th>30</th>
<th>26</th>
<th>14</th>
<th>28</th>
<th>13</th>
<th>25</th>
<th>14</th>
<th>27</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>24</td>
<td>16</td>
<td>21</td>
<td>16</td>
<td>15</td>
<td>22</td>
<td>21</td>
<td>17</td>
<td>21</td>
<td>29</td>
</tr>
<tr>
<td>26</td>
<td>21</td>
<td>15</td>
<td>20</td>
<td>30</td>
<td>24</td>
<td>20</td>
<td>20</td>
<td>15</td>
<td>26</td>
</tr>
<tr>
<td>29</td>
<td>30</td>
<td>20</td>
<td>27</td>
<td>22</td>
<td>26</td>
<td>22</td>
<td>28</td>
<td>30</td>
<td>15</td>
</tr>
</tbody>
</table>

Required:
A. Form a frequency table with sets to represent the data above.
B. Find the number of workers who got more than 20 days-off a year long.
UNIT THREE
Lesson Two

The Ascending and Descending Cumulative Frequency Table and Their Graphical Representation

Think and Discuss

You will learn how
- To Form both ascending and descending cumulative frequency tables.
- To represent both ascending and descending cumulative frequency tables graphically.

key terms
- The frequency distribution.
- The frequency table.
- The ascending cumulative frequency table.
- The descending cumulative frequency table.
- The ascending cumulative frequency curve.
- The descending cumulative frequency curve.

First: Ascending cumulative frequency table and its graphical representation.

Examples

The following table shows the frequency distribution for the heights of 100 students in a school in centimeters.

<table>
<thead>
<tr>
<th>Height (sets in cm)</th>
<th>115-</th>
<th>120-</th>
<th>125-</th>
<th>130-</th>
<th>135-</th>
<th>140-</th>
<th>145-</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of students (frequency)</td>
<td>8</td>
<td>12</td>
<td>19</td>
<td>23</td>
<td>18</td>
<td>13</td>
<td>7</td>
<td>100</td>
</tr>
</tbody>
</table>

1. How many students are with height less than 115 cm?
2. How many students are with height less than 135 cm?
3. How many students are with height less than 145 cm?

Form the ascending cumulative frequency table for these data and represent them graphically.

Solution

- Are there students with height less than 115 cm? **No**
- Are there students with height less than 135 cm? **How many?**
  - **yes, 62 student.**
- How can you calculate the number of students with height less than 145 cm? **Add the number of students in the sets of height less than the set 145.**

Now, to answer the previous questions in an easier way, form an ascending cumulative frequency table as follows:
### Upper boundaries of sets

<table>
<thead>
<tr>
<th>Upper boundaries of sets</th>
<th>Ascending cumulative frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Less than 115</td>
<td>0</td>
</tr>
<tr>
<td>Less than 120</td>
<td>0 + 8 = 8</td>
</tr>
<tr>
<td>Less than 125</td>
<td>8 + 12 = 20</td>
</tr>
<tr>
<td>Less than 130</td>
<td>20 + 19 = 39</td>
</tr>
<tr>
<td>Less than 135</td>
<td>39 + 23 = 62</td>
</tr>
<tr>
<td>Less than 140</td>
<td>82 + 18 = 80</td>
</tr>
<tr>
<td>Less than 145</td>
<td>80 + 13 = 93</td>
</tr>
<tr>
<td>Less than 150</td>
<td>93 + 7 = 100</td>
</tr>
</tbody>
</table>

### Ascending cumulative frequency table

<table>
<thead>
<tr>
<th>Upper boundaries of sets</th>
<th>Ascending cumulative frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Less than 115</td>
<td>zero</td>
</tr>
<tr>
<td>Less than 120</td>
<td>8</td>
</tr>
<tr>
<td>Less than 125</td>
<td>20</td>
</tr>
<tr>
<td>Less than 130</td>
<td>39</td>
</tr>
<tr>
<td>Less than 135</td>
<td>62</td>
</tr>
<tr>
<td>Less than 140</td>
<td>80</td>
</tr>
<tr>
<td>Less than 145</td>
<td>93</td>
</tr>
<tr>
<td>Less than 150</td>
<td>100</td>
</tr>
</tbody>
</table>

To represent the ascending cumulative frequency table graphically:

1. Specify the horizontal axis to the sets and the vertical axis to the ascending cumulative frequency.
2. Choose a drawing scale to draw the vertical axis such that the ascending cumulative frequency axis can hold the number of elements in a set.
3. Represent the ascending cumulative frequency for each set and draw its line graph successively.
Second: The descending cumulative frequency table and its graphical representation:

Of the previous frequency distribution which shows the heights of 100 students in a school in centimeters.

**Find:** The number of students with heights of 150cm and more.
- The number of students with heights of 140cm and more.
- The number of students with heights of 125cm and more.

Form the descending cumulative frequency table and represent it graphically.

**Solution**

There are no students with heights of 150cm and more.

The number of students with heights of 140cm and more is $7 + 13 = 20$ students.

The number of students with heights of 125cm and more is complete: $19 + \ldots + \ldots + \ldots + \ldots = \ldots$

To answer these questions in an easier way, form the descending cumulative frequency table as follows:

<table>
<thead>
<tr>
<th>Lower limits of sets</th>
<th>Ascending cumulative frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>115 and more</td>
<td>100</td>
</tr>
<tr>
<td>120 and more</td>
<td>92</td>
</tr>
<tr>
<td>125 and more</td>
<td>80</td>
</tr>
<tr>
<td>130 and more</td>
<td>61</td>
</tr>
<tr>
<td>135 and more</td>
<td>38</td>
</tr>
<tr>
<td>140 and more</td>
<td>20</td>
</tr>
<tr>
<td>145 and more</td>
<td>7</td>
</tr>
<tr>
<td>150 and more</td>
<td>zero</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Lower limits of sets</th>
<th>descending cumulative frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>115 and more</td>
<td>$92 + 8 = 100$</td>
</tr>
<tr>
<td>120 and more</td>
<td>$80 + 12 = 92$</td>
</tr>
<tr>
<td>125 and more</td>
<td>$61 + 19 = 80$</td>
</tr>
<tr>
<td>130 and more</td>
<td>$38 + 23 = 61$</td>
</tr>
<tr>
<td>135 and more</td>
<td>$20 + 18 = 38$</td>
</tr>
<tr>
<td>140 and more</td>
<td>$7 + 13 = 20$</td>
</tr>
<tr>
<td>145 and more</td>
<td>$0 + 7 = 7$</td>
</tr>
<tr>
<td>150 and more</td>
<td>$0$</td>
</tr>
</tbody>
</table>

To represent this table graphically, follow the steps of representing the ascending cumulative frequency to get the following graphical representation:
1. Below are the scores of 100 students in an experimental Maths exam.

<table>
<thead>
<tr>
<th>Sets</th>
<th>0–</th>
<th>10–</th>
<th>20–</th>
<th>30–</th>
<th>40–</th>
<th>50–</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>8</td>
<td>14</td>
<td>15</td>
<td>28</td>
<td>23</td>
<td>12</td>
<td>100</td>
</tr>
</tbody>
</table>

**Required:**

A. Form both the ascending and descending cumulative frequency tables.
B. Graph both the ascending and the descending cumulative frequency curves on the same graph paper.
C. From the graph, find the number of students who got less than 40 marks and those who got 40 marks and more.
D. Find the percentage of success, given that the minimum mark of success is 20 marks.
E. What is the percentage of the students who got more than 40 marks?

2. The following table shows the frequency distribution of the scores of 50 students in an experimental math exam.

<table>
<thead>
<tr>
<th>Intervalle</th>
<th>2–</th>
<th>6–</th>
<th>10–</th>
<th>14–</th>
<th>18–</th>
<th>22–</th>
<th>26–</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Effectif</td>
<td>3</td>
<td>5</td>
<td>9</td>
<td>10</td>
<td>12</td>
<td>7</td>
<td>4</td>
<td>50</td>
</tr>
</tbody>
</table>

**Required:** Graph the ascending cumulative frequency curve.
3. The following table shows the frequency distribution of the daily wages of some workers.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>10</td>
<td>14</td>
<td>24</td>
<td>30</td>
<td>12</td>
<td>10</td>
<td>100</td>
</tr>
</tbody>
</table>

Required: Graph the descending cumulative frequency curves.

4. The following table shows the frequency distribution of the ages of 50 workers.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>5</td>
<td>8</td>
<td>9</td>
<td>13</td>
<td>....</td>
<td>5</td>
<td>3</td>
<td>50</td>
</tr>
</tbody>
</table>

Required:
A. Complete the missing space.
B. Graph the ascending and descending cumulative frequency curves.
C. Find: First: the number of workers whose ages are more than 32.
   Second: The number of workers whose ages are less than 43.

5. The following table shows the frequency distribution of the scores of 1000 students in a final year exam.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of students</td>
<td>30</td>
<td>70</td>
<td>160</td>
<td>260</td>
<td>150</td>
<td>130</td>
<td>110</td>
<td>90</td>
<td>1000</td>
</tr>
</tbody>
</table>

Required:
A. Graph the ascending and descending cumulative frequency curves.
B. Find the number of students whose scores are less than 75 marks.
C. Find the number of students whose scores are more than 85 marks.
Arithmetic Mean, Median and Mode

Think and Discuss

First: the mean
You have learned to find the mean for a set of values and learned that:

\[
\text{The arithmetic mean} = \frac{\text{The sum of values}}{\text{Number of values}}
\]

Example: If the ages of 5 students are 13, 15, 16, 14, and 17 years old, then

The mean of their ages = \(\frac{13 + 15 + 16 + 14 + 17}{5}\)

= \(\frac{75}{5}\) = 15 years

Remark: \(15 \times 5 = 13 + 15 + 16 + 14 + 17\)

The mean is the simplest and most commonly used type of averages, it's that value given to each item in a set, then the total of these new values is the same total of the original values. It can be calculated by adding up all values, then divide the sum by the number of values.

Finding the mean of data from the frequency table with sets:
How can you find the mean of the following frequency distribution:

<table>
<thead>
<tr>
<th>Sets</th>
<th>10 –</th>
<th>20 –</th>
<th>30 –</th>
<th>40 –</th>
<th>50 –</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>10</td>
<td>20</td>
<td>25</td>
<td>30</td>
<td>15</td>
<td>100</td>
</tr>
</tbody>
</table>

Remark: To find the mean for a frequency distribution with sets, follow the following steps:
1. **Determine the centers of sets:**
   The center of the first set is \(\frac{20 + 10}{2} = 15\). The center of the second set is \(\frac{30 + 20}{2} = 25\) ... and so on.
   Since the ranges of the subsets are equal and each = 10
   We consider the upper limit of the last set = 60 and then:
   \[
   \text{its center} = \frac{50 + 60}{2} = 55
   \]

2. **Form the following vertical table:**

<table>
<thead>
<tr>
<th>Sets</th>
<th>Centre of the sets (X)</th>
<th>Frequency</th>
<th>Centre of the sets (\times) (X)</th>
<th>(\times) (F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 -</td>
<td>15</td>
<td>10</td>
<td>150</td>
<td></td>
</tr>
<tr>
<td>20 -</td>
<td>25</td>
<td>20</td>
<td>500</td>
<td></td>
</tr>
<tr>
<td>30 -</td>
<td>35</td>
<td>25</td>
<td>875</td>
<td></td>
</tr>
<tr>
<td>40 -</td>
<td>45</td>
<td>30</td>
<td>1350</td>
<td></td>
</tr>
<tr>
<td>50 -</td>
<td>55</td>
<td>15</td>
<td>825</td>
<td></td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>100</strong></td>
<td></td>
<td><strong>3700</strong></td>
<td><strong>3700</strong></td>
</tr>
</tbody>
</table>

3. **The mean** = \[
   \frac{\text{The total of } (F \times X)}{\text{the total of } F}
   \]

   \[
   = \frac{3700}{100} = 37
   \]

**Practice**

1. If the mean of the scores of a student during the first 5 months is 23.8. What is the score of the 6th month if the mean of his scores is 24 marks?
2. The following table shows the frequency distribution of the weights of 30 children in kg.

<table>
<thead>
<tr>
<th>Weight in (kg)</th>
<th>6–</th>
<th>10–</th>
<th>14–</th>
<th>18–</th>
<th>22–</th>
<th>26–</th>
<th>30–</th>
<th><strong>Total</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>frequency</td>
<td>2</td>
<td>3</td>
<td>....</td>
<td>8</td>
<td>6</td>
<td>4</td>
<td>2</td>
<td><strong>30</strong></td>
</tr>
</tbody>
</table>

Complete the table, then find the mean of such a distribution.
Second: the median

The median is the middle value in a set of values after arranging it ascendingly or descendingly such that the number of values which are less than it is equal to the number of values which are greater than it.

Finding the median of a frequency distribution with sets graphically:

1. Draw the ascending or descending cumulative frequency table, then draw the cumulative frequency curve of it.
2. Determine the order of the median = \( \frac{\text{The total of frequency}}{2} \).
3. Determine point A on the vertical axis (frequency) which represents the order of the median.
4. Draw a horizontal straight line from point A to intersect the curve at a point. Form this point, draw a vertical straight line on the horizontal axis to intersect it at a point that represents the median.

Example (1)

The following table shows the frequency distribution for the scores of 60 students in an exam.

<table>
<thead>
<tr>
<th>Sets</th>
<th>2–</th>
<th>6–</th>
<th>10–</th>
<th>14–</th>
<th>18–</th>
<th>22–</th>
<th>26–</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>6</td>
<td>9</td>
<td>12</td>
<td>15</td>
<td>10</td>
<td>5</td>
<td>3</td>
<td>60</td>
</tr>
</tbody>
</table>

Find the median of the distribution using the ascending cumulative frequency table.

Solution

1. Draw an ascending cumulative frequency table.
2. Find the order of the median = \( \frac{60}{2} = 30 \)
3. Draw the ascending cumulative frequency curve, and get the median form the graph.

From the graph, the median = 14.8 mark
Think up Can you find the median using the descending cumulative frequency table? Is the value of the median different in such a case?

Example (2)

The following table shows the daily wages of 100 workers in a factory.

<table>
<thead>
<tr>
<th>daily wages in LE (sets)</th>
<th>15−</th>
<th>20−</th>
<th>25−</th>
<th>30−</th>
<th>35−</th>
<th>40−</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of workers (frequency)</td>
<td>10</td>
<td>15</td>
<td>22</td>
<td>25</td>
<td>20</td>
<td>8</td>
<td>100</td>
</tr>
</tbody>
</table>

Required:
1. Graph the ascending and descending cumulative frequency curves on one figure.
2. Can you find the median wage from this curve?

Solution

<table>
<thead>
<tr>
<th>Upper boundaries of sets</th>
<th>Cumulative frequency</th>
<th>Lower boundaries of sets</th>
<th>Cumulative frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Less than 15</td>
<td>zero</td>
<td>15 and more</td>
<td>100</td>
</tr>
<tr>
<td>Less than 20</td>
<td>10</td>
<td>15 and more</td>
<td>90</td>
</tr>
<tr>
<td>Less than 25</td>
<td>25</td>
<td>15 and more</td>
<td>75</td>
</tr>
<tr>
<td>Less than 30</td>
<td>47</td>
<td>15 and more</td>
<td>53</td>
</tr>
<tr>
<td>Less than 35</td>
<td>72</td>
<td>15 and more</td>
<td>28</td>
</tr>
<tr>
<td>Less than 40</td>
<td>92</td>
<td>15 and more</td>
<td>8</td>
</tr>
<tr>
<td>Less than 45</td>
<td>100</td>
<td>15 and more</td>
<td>zero</td>
</tr>
</tbody>
</table>

Remark:
The ascending cumulative frequency curve intersects with the descending cumulative frequency curve at one point which is \( m \).
The y-coordinate for the point $M = 50$ 
\[ = \frac{100}{2} \]

= the order of the median

\[ \therefore \text{The X-coordinate of the point M determines the median} \]

every 10 mm of the x coordinate represents LE 5

**Complete:** 2 mm represents ........
The median wage = 30 + \( \frac{2 \times 5}{10} \) = LE 31.

**practice**

**Draw** the descending cumulative frequency curve for the following frequency distribution, then find the value of the median.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>4</td>
<td>6</td>
<td>10</td>
<td>17</td>
<td>10</td>
<td>3</td>
<td>50</td>
</tr>
</tbody>
</table>

**Third: the mode**

The mode is the most common value in the set or in other words, it is the value which is repeated more than any other values.

**Example**

The following table shows the frequency distribution for the scores of 40 students in an examination.

<table>
<thead>
<tr>
<th>Sets</th>
<th>2–</th>
<th>6–</th>
<th>10–</th>
<th>14–</th>
<th>18–</th>
<th>22–</th>
<th>26–</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>10</td>
<td>7</td>
<td>5</td>
<td>2</td>
</tr>
</tbody>
</table>

Find the mode of this distribution graphically

**Solution**

You can find the mode of this distribution graphically using the histogram as follows:

1. First: draw a histogram.

Draw two perpendicular axes: one horizontal to represent sets and the other vertical to represent the frequency of each set.
2. Divide the horizontal axis into a number of equal parts using a suitable drawing scale to represent sets.
3. Divide the vertical axis into a number of equal parts using a suitable drawing scale such that the greatest frequency among sets can be represented.
4. Draw a rectangle whose base is set (2-) and height is equal to the frequency (3).
5. Draw another rectangle adjacent to the first one whose base is set (6-) and height is equal to the frequency (5).
6. Repeat drawing the rest of adjacent rectangles till the last set (26-).

Second: Finding the mode from the histogram, to find the mode from the histogram, we observe that: the most repeated set is (14-), and it is called the mode set, why?

Define the intersection point of $\overline{AD}$, $\overline{BC}$ from the graph, and from this point, drop a vertical line on the horizontal axis to define the sequential value within that distribution.

From the graph, what's the mode value?

---

**Exercises (3 – 3)**

1. The following table shows the frequency distribution of 50 workers days-off:

<table>
<thead>
<tr>
<th>Sets</th>
<th>2-</th>
<th>6-</th>
<th>10-</th>
<th>14-</th>
<th>18-</th>
<th>22-</th>
<th>26-</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>4</td>
<td>5</td>
<td>8</td>
<td>K-2</td>
<td>7</td>
<td>5</td>
<td>1</td>
</tr>
</tbody>
</table>

**Find**

- A. the value of K
- B. The Arithmetic mean for that distribution.

2. The following table shows the frequency distribution of the heights of 120 students in centimeters:

<table>
<thead>
<tr>
<th>Height in (cm)</th>
<th>140-</th>
<th>144-</th>
<th>148-</th>
<th>152-</th>
<th>156-</th>
<th>160-</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>12</td>
<td>20</td>
<td>38</td>
<td>22</td>
<td>17</td>
<td>11</td>
<td>120</td>
</tr>
</tbody>
</table>

Find the mean.
3. The following table shows the frequency distribution of 50 workers’ wages in a factory:

<table>
<thead>
<tr>
<th>sets of wages</th>
<th>300-</th>
<th>400-</th>
<th>500-</th>
<th>600-</th>
<th>700-</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of workers</td>
<td>8</td>
<td>12</td>
<td>18</td>
<td>7</td>
<td>5</td>
<td>50</td>
</tr>
</tbody>
</table>

Graph the descending cumulative frequency curve, then find the median.

4. From the following frequency table with equal sets in range.

<table>
<thead>
<tr>
<th>Sets</th>
<th>10-</th>
<th>20-</th>
<th>30-</th>
<th>40-</th>
<th>x-</th>
<th>60-</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>12</td>
<td>15</td>
<td>25</td>
<td>27</td>
<td>k+4</td>
<td>4</td>
<td>100</td>
</tr>
</tbody>
</table>

A. The value of both X and K.
B. Graph the ascending and descending cumulative Frequency curves in one figure, then find the median.

5. The following table shows the frequency distribution of the weights of 50 students in Kg.

<table>
<thead>
<tr>
<th>weight in kg</th>
<th>30-</th>
<th>35-</th>
<th>40-</th>
<th>45-</th>
<th>50-</th>
<th>55-</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of students</td>
<td>k+4</td>
<td>3k</td>
<td>4k</td>
<td>3k+1</td>
<td>3k-1</td>
<td>k+1</td>
<td>50</td>
</tr>
</tbody>
</table>

Find

A. the value of K.
B. Graph the frequency histogram, then find the mode weight.

6. The following table shows the frequency distribution of the height of 200 students.

<table>
<thead>
<tr>
<th>Height in (cm)</th>
<th>110-</th>
<th>115-</th>
<th>120-</th>
<th>125-</th>
<th>130-</th>
<th>135-</th>
<th>140-</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of students</td>
<td>10</td>
<td>12</td>
<td>28</td>
<td>35</td>
<td>60</td>
<td>40</td>
<td>40</td>
<td>200</td>
</tr>
</tbody>
</table>

Graph the frequency histogram, then find the mode length.
General Exercises

1. The following table shows the frequency distribution for the scores of 50 students in an examination:

<table>
<thead>
<tr>
<th>Sets</th>
<th>2-</th>
<th>6-</th>
<th>10-</th>
<th>14-</th>
<th>18-</th>
<th>22-</th>
<th>26-</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>3</td>
<td>5</td>
<td>9</td>
<td>10</td>
<td>12</td>
<td>7</td>
<td>4</td>
<td>50</td>
</tr>
</tbody>
</table>

Find First: the mean of the student's score. Second: The median

2. From the following frequency table with equal sets in range, find:

<table>
<thead>
<tr>
<th>Sets</th>
<th>10-</th>
<th>20-</th>
<th>x-</th>
<th>40-</th>
<th>50-</th>
<th>60-</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>10</td>
<td>17</td>
<td>20</td>
<td>32</td>
<td>K + 2</td>
<td>4</td>
<td>100</td>
</tr>
</tbody>
</table>

First: find the value of X and K:
Second: graph the ascending and descending cumulative curves on one figure, then calculate the median

3. Find the mode of the following frequency distribution for the scores of 40 student in an examination:

<table>
<thead>
<tr>
<th>Sets of marks</th>
<th>30-</th>
<th>40-</th>
<th>50-</th>
<th>60-</th>
<th>70-</th>
<th>80-</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>frequency</td>
<td>3</td>
<td>4</td>
<td>12</td>
<td>8</td>
<td>7</td>
<td>6</td>
<td>40</td>
</tr>
</tbody>
</table>

4. The following table shows the frequency distribution with equal-range sets for the weekly wages of 100 works in a factory.

<table>
<thead>
<tr>
<th>Sets of wages in L.E</th>
<th>70-</th>
<th>80-</th>
<th>90-</th>
<th>100-</th>
<th>x-</th>
<th>120-</th>
<th>130-</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of workers</td>
<td>10</td>
<td>13</td>
<td>f - 4</td>
<td>20</td>
<td>16</td>
<td>14</td>
<td>11</td>
</tr>
</tbody>
</table>

Trouver A The value of x and F
            B The mode of wages in L.E
Activity

The following table shows the frequency distribution for the weights of 50 students in K.g at a school.

<table>
<thead>
<tr>
<th>Weight in K.g</th>
<th>30–</th>
<th>35–</th>
<th>40–</th>
<th>45–</th>
<th>50–</th>
<th>55–</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number. of students</td>
<td>7</td>
<td>3k</td>
<td>4k</td>
<td>10</td>
<td>8</td>
<td>4</td>
<td>50</td>
</tr>
</tbody>
</table>

First: find the value of K.
Second: calculate the mean.
Third: Draw the ascending cumulative frequency curve.
Fourth: Draw the histogram and find the mode of weights.
Fifth: Find the median.
1. Complete the following:
   A. If the lower limit of a set is 8 and the upper limit of the same set is 14, then its centre is ........
   B. If the lower limit of a set is 4 and its centre is 9, then its upper limit is ........
   C. The intersection point of the ascending and descending cumulative frequency curves determines ........ on the set's axis.
   D. The location of the top of the frequency curve on the set axis is ........
   E. If the mean of a frequency distribution is 39.4 and the total of its frequency is 100, then the total of the product of multiplying each set frequency by its centre = ........

2. The following table shows the frequency distribution of weights of 20 children in k.g.

<table>
<thead>
<tr>
<th>sets</th>
<th>5 –</th>
<th>15 –</th>
<th>25 –</th>
<th>35 –</th>
<th>45 –</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>3</td>
<td>4</td>
<td>7</td>
<td>4</td>
<td>2</td>
<td>20</td>
</tr>
</tbody>
</table>

Find the median weight in k.g. using the ascending and descending cumulative frequency curve of this distribution.

3. Below is the frequency distribution of the weekly bonus of 100 workers in a factory.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of workers</td>
<td>10</td>
<td>K</td>
<td>22</td>
<td>26</td>
<td>20</td>
<td>8</td>
</tr>
</tbody>
</table>

A. Calculate the value of K.
B. Find the mean of this distribution.
C. The mode value of the weekly bonus using the histogram..
UNIT FOUR

4

Geometry
You will learn how
- Medians of the triangle.
- A 30° - 60° - 90° triangle.

**Key terms**
- Median of the triangle.
- A 30° - 60° - 90° triangle.
- Point of Concurrence

**Think and Discuss**

The medians of a triangle is the line segment drawn from the triangle vertex to the middle of the opposite side of this vertex.

ABC is a triangle where the point D bisects BC.

So AD is a triangle Median.

- How many medians does the triangle have?
- Draw the medians in each triangle.

**Theorem 1**

The medians of a triangle are concurrent

ABC is a triangle where point D bisects BC, point E bisects AC, point F bisects AB, then AD, BE and CF all intersect in one point (M).

**Practice**

In the figure opposite: ABC is a triangle where point x bisects BC, point y bisects AB, and AX ∩ CY = {N}.
1. Draw $\overrightarrow{BN}$ to intersect $\overline{AC}$ at point $Z$, then find the lengths of $\overline{AZ}$ and $\overline{CZ}$ is $AZ = CZ$? Reason your answer.

2. Measure, then complete.

$$\frac{NX}{NA} = \ldots, \quad \frac{NY}{NC} = \ldots, \quad \frac{NZ}{NB} = \ldots$$

- If your measurements are accurate, then $\frac{NX}{NA} = \frac{1}{2}, \quad \frac{NY}{NC} = \frac{1}{2}$ and $\frac{NZ}{NB} = \frac{1}{2}$

**Theorem 2**

The point of concurrence of the medians of the triangle divides each median in the ratio of 1:2 from its base.

**Practice**

**Complete**

- $\overline{ME} = 3\text{cm}, \quad \overline{MC} = 8\text{cm}$
- $\overline{MA} = \ldots, \quad \overline{MD} = \ldots$
- $\overline{ME} = \ldots, \quad \overline{AE}, \quad \overline{MC} = \ldots, \quad \overline{CD}$
- $\overline{LZ} = 15\text{cm}, \quad \overline{YM} = 18\text{cm}, \quad \overline{XY} = 20\text{cm}$
- $\overline{NL} = \ldots, \quad \overline{NY} = \ldots$
- Perimeter of $\triangle NLY = \ldots$

**Fact**

$\overline{AD}$ is a median in $\triangle ABC$, $M \in \overline{AD}$

if $\overline{AM} = 2\overline{MD}$,

then

then **$M$ is the point where the medians of the triangle intersect.**
Example (1)

In the figure opposite: ABCD is a parallelogram where its two diagonals intersect at point M, point \( E \in DM \) and \( DE = 2EM \).

\( CE \) is drawn and intersected \( AD \) at point \( F \).

Prove that: \( AF = FD \)

Proof: In the parallelogram \( ABCD \)
\[ \therefore AC \cap BD = \{M\} \]
\[ \therefore M \text{ bisects } AC \]
the triangle \( DAC \)
\[ \therefore M \text{ bisects } AC \]
\[ \therefore DM \text{ is a median of the triangle.} \]
\[ \therefore E \in DM, \ DE = 2EM. \]
\[ \therefore E \text{ is the intersecting point of the triangle's medians.} \]
\[ \therefore E \in CF \]
\[ \therefore CF \text{ is a median of the triangle and point } F \text{ bisects } AD \]

Theorem 3

In the right - angled triangle, the length of the median from the vertex of the right angle equals half the length of the hypotenuse.

Given data: \( ABC \) is a triangle where \( m(\angle B) = 90^\circ \),
\( BD \) is a median in \( \triangle ABC \).

Required: Prove that: \( BD = \frac{1}{2} AC \).

Construction : Draw \( \overrightarrow{BD} \), let point \( E \in \overrightarrow{BD} \) where \( BD = DE \).

Proof :
\[ \therefore \text{in the Figure } ABCE, \ AC, \ BE \text{ bisect each other.} \]
the Figure ABCE is a parallelogram.

\[ m(\angle B) = 90^\circ \]

\( \therefore \) ABCE is a rectangle.

\[ BE = AC. \]

\[ \therefore BD = \frac{1}{2} BE \]

\[ \therefore BD = \frac{1}{2} AC \]

Q.E.D.

### The converse of the theorem 3

If the length of the median drawn from a vertex of a triangle equals half the length of the opposite side to this vertex, then the angle at this vertex is right.

**Given:** ABC is a triangle where \( \overline{BD} \) is a median,

\[ BD = DA = DC \]

**Required:** Prove that: \( m(\angle ABC) = 90^\circ \).

**Construction:** Draw \( \overrightarrow{BD} \) and let point \( E \in \overrightarrow{BD} \)

where \( BD = DE \).

**Proof:** \( \therefore BD = \frac{1}{2} BE = \frac{1}{2} AC. \)

\[ \therefore BE = AC. \]

In the Figure ABCE, AC and BE are equal in length and bisect each other.

\( \therefore \) the Figure ABCE is a rectangle.

\[ \therefore m(\angle ABC) = 90^\circ \]

Q.E.D.

### Corollary

The length of the side opposite to the angle of measure 30° in the right-angled triangle equals half the length of the hypotenuse.

**Remember:**

If the point D bisects \( \overline{AB} \) and the point E bisects \( \overline{AC} \), Then:

1. \[ \overline{DE} = \frac{1}{2} \overline{BC} \]
2. \[ \overline{DE} \parallel \overline{BC} \]
Exercises (4-1)

Complete:

1. BD = ....... cm, AB = ....... cm
   Perimeter $\triangle ABD = ....... cm$

2. AC = ....... cm, BD = ....... cm
   MD = ....... BD, MD = ....... cm

3. DF = ....... cm, DE = ....... cm,
   FE = ....... cm
   Perimeter $\triangle DEF = ....... cm$

4. AC = ....... cm, AD = ....... cm
   BC = ....... cm, CD = ....... cm

5. In the figure opposite:
   ABC is a triangle, X bisect $\overline{AB}$,
   Y bisect $\overline{BC}$, $XY = 5$ cm,
   $X C \cap A Y = \{M\}$
   where: $CM = 8cm$, $YM = 3$ cm

Find:

(1) The perimeter of $\triangle MXY$
(2) The perimeter of $\triangle MAC$
6. **ABC** is a triangle where point **D** bisects \( BC \), and point \( M \in AD \). \( AM = 2 MD \).

   Draw \( CM \) to intersect \( AB \) at point **E**.

   If \( EC = 12 \text{ cm} \), then find the length of \( EM \).

7. **In figure opposite:**

   **ABC** is a right-angled triangle at **B**,
   \[ m(\angle ACB) = 30^\circ \]
   \[ AB = 5 \text{ cm}, \text{ point } E \text{ is the mid point of } AC. \]
   If \( DE = 5 \text{ cm} \), then prove that \( m(\angle ADC) = 90^\circ \).
Think and Discuss

You have learnt that triangles are classified according to the lengths of their sides into three types:

<table>
<thead>
<tr>
<th>The scalene triangle</th>
<th>The isosceles triangle (two sides are congruent)</th>
<th>The equilateral triangle (three sides are congruent)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A B ≠ B C</td>
<td>A B = A C</td>
<td>A B = A C = B C</td>
</tr>
<tr>
<td>A B ≠ A C</td>
<td></td>
<td></td>
</tr>
<tr>
<td>B C ≠ A C</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In the figure opposite:

**Remark:** the two sides \( A B, A C \) are congruent (of equal lengths), so the triangle ABC is called isosceles triangle while point A is called the vertex. \( B C \) is the base, and the two angles B and C are the base angles of the triangle.

**The properties of isosceles triangle**

In any isosceles triangle

- What is the type of the base angles? (acute - right - obtuse)
- What is the type of the vertex angle?
Practice

In each of the following figures, state the isosceles triangles and define their bases, then notice the type of the two base angles and the vertex angle.

1. A
   B
   C
   3 Cm
   4 Cm

2. A
   B
   D
   E
   C

3. A
   D
   E
   F
   3 Cm
   4 Cm

4. L
   M
   N
   X
   Y
   Z
   L
   M
   N
   X
   Y
   Z

5. A
   B
   C
   D
   5 Cm
   4 Cm

6. X
   Y
   Z
   L
   M
   N
   X
   Y
   Z
   L
   M
   N

7. X
   Y
   Z
   L
   M
   N
   X
   Y
   Z
   L
   M
   N

8. A
   B
   C
   D
   4 Cm
   5 Cm

9. A
   B
   C
   D
   6 Cm
   5 Cm

Remark:
1. Both of the base angles in the isosceles triangle are acute.
2. The vertex angle in the isosceles triangle can be either acute, right or obtuse.
   So, the isosceles triangle can be either obtuse, right or acute angled triangle as shown in the following chart:
Think and Discuss

Is there a relation among the measures of the two base angles in the isosceles triangle?

to know that, let’s conduct the next activity:

**Activity**

**Using the compass**
1. Draw several isosceles triangles as shown in the opposite figure.
   Where \( AB = AC \).
2. **Find** using a protractor, the measure of the two base angles \( \angle ABC \) and \( \angle ACB \).
3. Write down the data you got in a table as follows, then compare the measures in each case.

<table>
<thead>
<tr>
<th>Number of the triangle</th>
<th>( m(\angle ABC) )</th>
<th>( m(\angle ACB) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4. Keep your activity in the portfolio.

**Theorem 1**

(the isosceles triangle theorem) the base angles of the isosceles triangle are congruent.

**Given:** \( \triangle ABC \) is triangle in which \( AB = AC \)

**R.T.P:** \( \angle B = \angle C \)
Construction: draw \( \overline{AD} \perp \overline{BC} \)

Proof: The two triangles ADB and ADC are right angled in which:

\[
\begin{align*}
\frac{AB}{AD} &= \frac{AC}{AD} \\
\therefore \triangle ADB &= \triangle ADC \\
(a \text{ given}) \\
(a \text{ common side}) \\
(a \text{ hypotenuse and a side}) 
\end{align*}
\]

from the congruency, we deduce that \( \angle B = \angle C \)

Q.E.D.

practice

1. In each of the following figures, find the value of the symbol that is used to measure the angle:

   - \( x = \ldots \ldots \)
   - \( y = \ldots \ldots \)
   - \( N = \ldots \ldots \)
   - \( E = \ldots \ldots \)
   - \( x = \ldots \ldots, y = \ldots \ldots, L = \ldots \ldots \)
   - \( L = \ldots \ldots, M = \ldots \ldots \)
   - \( L = \ldots \ldots, I = \ldots \ldots \)
   - \( y = \ldots \ldots, L = \ldots \ldots, I = \ldots \ldots \)
2. In the figure opposite, \( \Delta ABC \) is an isosceles triangle in which \( AB = AC \)

\[ m(\angle A) = 40^\circ, \quad D \in \overline{CB}, \quad E \in \overline{BC}. \]

**First:** Find \( m(\angle ABC) \)

**Second:** Prove that \( \angle ABD = \angle ACE \)

**Think:** Are the supplementary angles to congruent angles congruent?

**Corollary**

If the triangle is equilateral, then it is equiangular where each angle measure 60°.

**Example (1)**

In the figure opposite: \( \Delta ABC \) is an equilateral triangle.

\( D \in \overline{BC} \) such that \( BC = CD \)

prove that \( \frac{AB}{AD} = \frac{BA}{AD} \)

**Given:** \( AB = BC = CA = CD, \quad D \in \overline{BC} \)

**R.T.P.:** Prove that \( \frac{BA}{AD} = \frac{AD}{DA} \)

**Proof:** \( \therefore \Delta ABC \) is an equilateral triangle.

\[ \therefore m(\angle ACB) = m(\angle BAC) = m(\angle B) = 60^\circ \quad \text{(corollary)} \]

\[ \therefore D \in \overline{BC} \]

\[ \therefore \angle BCA \text{ is an exterior angle of the \( \Delta ACD \)} \]

\[ m(\angle BCA) = m(\angle CAD) + m(\angle CDA) = 60^\circ \quad (1) \]

In \( \Delta ACD \)

\[ \therefore CA = CD \quad \therefore m(\angle CAD) = m(\angle CDA) \quad (2) \]

from (1), (2) we deduce that: \( m(\angle CAD) = m(\angle CDA) = 30^\circ \)
\[ \therefore m(\angle BAD) = m(\angle BAC) + m(\angle CAD) \]
\[ \therefore m(\angle BAD) = 60^\circ + 30^\circ = 90^\circ \]
\[ \therefore \quad \frac{BA}{AD} \quad Q.E.D \]

**Remark:** The measure of an exterior angle of a triangle is equal to the sum of the measures of the two non-adjacent interior angles.

**Example (2)**

- **In the figure opposite:** AB = AD, BC = CD

- **Prove that** \( \angle ABC = \angle ADC \)

**Given:** AB = AD, BC = CD

**R.T.P:** prove that \( \angle ABC = \angle ADC \)

**Proof:** In \( \triangle ABD \)

\[ \therefore AB = AD \]
\[ \therefore m(\angle ABD) = m(\angle ADB) \quad (1) \]

in \( \triangle CBD \)

\[ \therefore CB = CD \]
\[ \therefore m(\angle CBD) = m(\angle CDB) \quad (2) \]

By adding (1) and (2) we deduce that

\[ m(\angle ABD) + m(\angle CBD) = m(\angle ADB) + m(\angle CDB) \]

\[ \therefore m(\angle ABC) = m(\angle ADC) \]

\[ \angle ABC = \angle ADC \quad Q.E.D. \]
In each of the following figures, find the value of the symbol that is used to measure the angle:

1. \( x = \ldots \), \( y = \ldots \)

2. \( x = \ldots \), \( y = \ldots \)

3. \( x = \ldots \), \( y = \ldots \)

4. \( \overline{AD} \parallel \overline{BC} \)

5. \( \overline{AE} \parallel \overline{BC} \)

6. \( \overline{AE} \) Bisects \( \angle CAD \)

\[ \begin{array}{|c|c|c|c|c|c|} 
\hline
\text{Number of the triangle} & \text{BC} & m(\angle B) & m(\angle C) & \text{AB} & \text{AC} \\
\hline
1 & 7 \text{cm} & 50^\circ & 50^\circ & \ldots \ldots & \ldots \ldots \\
2 & \ldots \ldots & 50^\circ & 50^\circ & \ldots \ldots & \ldots \ldots \\
3 & \ldots \ldots & \ldots \ldots & \ldots \ldots & \ldots \ldots & \ldots \ldots \\
4 & \ldots \ldots & \ldots \ldots & \ldots \ldots & \ldots \ldots & \ldots \ldots \\
\hline
\end{array} \]

1. Are \( \overline{AB} \) and \( \overline{AC} \) equal in length?

2. Is \( \overline{AB} \equiv \overline{AC} \)?

3. How can you explain such corollaries geometrically?
Theorem (2)

If two angles of a triangle are congruent, then the two sides opposite to these two angles are congruent and the triangle is isosceles.

\[ \triangle ABC, \angle B = \angle C \]

R.T.P: Prove that: \[ AB = AC \]

Construction: bisect \( \angle BAC \) with the bisector \( AD \) to intersect \( BC \) at \( D \)

Proof:
\[ \therefore \angle B \equiv \angle C \]
\[ \therefore m(\angle B) = m(\angle C) \]
\[ \therefore \frac{AD}{BC} \text{ bisects } \angle BAC \]
\[ \therefore m(\angle BAD) = m(\angle CAD) \]
\[ \therefore \text{ the sum of the measures of interior angles of a triangle is } \approx 180^\circ \]
\[ \therefore m(\angle ADB) = m(\angle ADC) \]
\[ \therefore \text{ In the two triangles } ADB, ADC \]

\[ \overline{AD} \text{ is a common side} \]
\[ m(\angle BAD) = m(\angle CAD) \]
\[ m(\angle ADB) = m(\angle ADC) \]
\[ \therefore \triangle ADB \equiv \triangle ADC \]

Form the congruency, we deduce that \[ AB \equiv AC \]

Therefore, \( \triangle ABC \) is an isosceles triangle

Corollary

If the angles of a triangle are congruent, then the triangle is equilateral.

In the figure opposite \( \triangle ABC \) is an isosceles triangle in which:
\[ AB = AC, m(\angle BAC) = 60^\circ \]

Complete \[ m(\angle ABC) = m(\angle ACB) = \ldots \]
\[ i.e: \angle \ldots = \angle \ldots = \angle \ldots \]
\[ \therefore \angle \triangle ABC \text{ is } \ldots \triangle \]
Remark: In an isosceles triangle, if any angle has a measure of 60°, then the triangle is an equilateral triangle.

practice

In each of the following figures, define the triangle’s sides that are equal in length as shown in example 1:

1. AB = AC

2. ............ = ............

3. ............ = ............

4. ............ = ............

5. ............ = ............

6. ............ = ............

7. ............ = ............

8. ............ = ............

9. ............ = ............

(Images of triangles with labeled sides and angles are shown.)
Examples

1. **In the figure opposite:** ABC is a triangle in which AB = AC, \( \overline{XY} \parallel \overline{BC} \)

   **prove that** \( \triangle AXY \) is an isosceles triangle

**Given:** \( AB = CA, \overline{XY} \parallel \overline{BC} \).

**Required:** prove that \( AX = AY \)

**Proof:** In \( \triangle ABC \) \( \angle A = \angle C \)

\[ \therefore m(\angle ABC) = m(\angle ACB) \quad (1) \]

\[ \therefore \overline{XY} \parallel \overline{BC}, \text{ AB a transversal} \]

\[ \therefore m(\angle AXY) = m(\angle ABC) \text{ correspondingly} \quad (2) \]

The same \( \overline{XY} \parallel \overline{BC}, \text{ AC a transversal} \)

\[ \therefore m(\angle AYX) = m(\angle ACB) \text{ correspondingly} \quad (3) \]

From (1), (2), (3) we deduce that:

\[ m(\angle AXY) = m(\angle AYX) \]

In \( \triangle AXY \)

\[ \therefore m(\angle AXY) = m(\angle AYX) \]

\[ \therefore AX = AY \]

i.e. the triangle AXY is an isosceles triangle \( \text{Q.E.D} \)

**Think:** Can we deduce that XA = YA? Explain your answer,
2. In the figure opposite:

ABC is a right angled triangle at B, \( m(\angle C) = 30^\circ \),

\( D \in \overline{AC} \) where \( DB = DC \)

**prove that** \( \triangle ABD \) is an equilateral triangle.

**Given:** \( m(\angle A B C) = 90^\circ \), \( m(\angle C) = 30^\circ \), \( DB = DC \)

**R.T.P:** prove that \( AB = BD = AD \)

**proof:** In \( \triangle DBC \) \( \therefore DB = DC \)

\[ \because m(\angle DBC) = m(\angle C) = 30^\circ \]

In \( \triangle ABC \) \( \because m(\angle ABC) = 90^\circ \), \( m(\angle DBC) = 30^\circ \)

\[ \therefore m(\angle BAD) = 90 - 30 = 60^\circ \] \( (1) \)

\( \because \angle ADB \) is an exterior angle of \( \triangle BDC \)

\[ \therefore m(\angle ADB) = m(\angle DBC) + m(\angle DCB) \]

\[ m(\angle ADB) = 30^\circ + 30^\circ = 60^\circ \] \( (2) \)

In \( \triangle ABD \) \( \therefore \) the sum of the measures of the interior angles of a triangle = \( 180^\circ \)

\[ \therefore m(\angle ABD) = 180^\circ - (60^\circ + 60^\circ) = 60^\circ \] \( (3) \)

From (1), (2), (3) \( \therefore m(\angle ABD) = m(\angle ADB) = m(\angle A) \)

**i.e.** \( \angle ABD = \angle ADB = \angle A \)

\( \therefore \) the triangle \( ABD \) is equilateral \( \quad \text{i.e.} \quad AB = BD = AD \)
Exercises (4-3)

1. Complete:
   \[ \triangle ABC \]
   \[ AC = \ldots \text{ cm} \]
   \[ XZ = \ldots \text{ cm} \]
   \[ DE = \ldots \text{ cm}, \ m(\angle E) = \ldots^\circ \]
   \[ EF = \ldots \text{ cm}, \ m(\angle MDF) = \ldots^\circ \]

2. In the figure opposite:
   \[ AB = AC, \ m(\angle BAC) = 48^\circ \]
   \[ CD \text{ bisects } \angle BCA \text{ and intersects } AB \text{ at } D \]
   Find \[ m(\angle B) \text{ et } m(\angle BCD) \]

3. In the figure opposite:
   \[ \triangle ABC \]
   \[ AD \parallel BC, \ m(\angle DAC) = 30^\circ \]
   Find The measures of the angles in \[ \triangle ABC \]

4. In the figure opposite:
   \[ \triangle XYZ \]
   \[ Z \dot{\in} \overline{LY}, \ XZ = YZ \]
   \[ m(\angle LXZ) = 130^\circ, \ \overline{LM} \parallel \overline{XY} \]
   Find \[ m(\angle MLY) \]
5. In the figure opposite:
   \[ AB = AC, \quad m(\angle B) = (2x + 13)^\circ \]
   \[ m(\angle C) = (3x - 17)^\circ \]

Prove that: \( \triangle ADE \) is an isosceles triangle.

6. In the figure opposite:
   \( \triangle ABC \) is an isosceles triangle in which \( AB = AC \),
   \( D \in BC \), \( E \in BC \) such that \( BD = EC \)

Prove that: \( \triangle ADE \) is an isosceles triangle.

7. In the figure opposite: \( \triangle ABC \) is an equilateral triangle.
   \( E \in AC \), \( D \in CB \),
   \[ m(\angle DEC) = 30^\circ \]

Prove that \( \triangle DCE \) is an isosceles triangle.

8. In the figure opposite:
   \( BD \) bisects \( \angle ABC \) and intersects \( AC \) at \( D \),
   \( DE \parallel BC \) where \( E \in AB \).

Prove that \( \triangle EBD \) is an isosceles triangle.

9. \( \triangle ABC \) is a triangle in which \( D \in AB \),
   \( E \in BC \) such that \( BD = BE \), So if \( DE \parallel AC \)

Prove that \( AB = BC \)

10. \( \triangle ABC \) is a triangle in which \( AB = AC \),
    \( BD \) bisects \( \angle ABC \), \( CD \) bisects \( \angle ACB \), \( BD \cap CD = \{ D \} \)

Prove that \( \triangle DB \) is an isosceles triangle.
ABCD is a square in which its diagonals $AC$, $BD$ intersect at $M$.

**Complete and discuss.**

**A** In $\triangle ABC$, $m(\angle ABC) =$ \[ \ldots \] 
$\therefore$ $AB = BC$ 
$\therefore m(\angle BAC) = m(\angle BCA) =$ \[ \ldots \] 

**B** $m(\angle BAD) = 90^\circ$ 
$\therefore m(\angle DAC) =$ \[ \ldots \] 
$\therefore m(\angle BCD) = 90^\circ$ 
$\therefore m(\angle ACD) =$ \[ \ldots \] 

**C** Does the diagonal $AC$ bisect $\angle A$?

**D** Does the diagonal $BD$ bisect both $\angle B$ and $\angle D$?

**E** Is $\triangle MAD$ an isosceles triangle? Why?

**F** State those isosceles triangles whose vertices are point $M$.

**G** Is $M$ the midpoint of $AC$ and $BD$?

**H** Is $AC = BD$?

**I** From the previous clauses, deduce the properties of the square and keep them in
Think and Discuss

Corollary (1)

The median of an isosceles triangle from the vertex angle bisects it and is perpendicular to the base.

In the figure opposite:

In \( \triangle ABC \), \( AB = AC \), \( \overrightarrow{AD} \) is a median

then: \( \overrightarrow{AD} \) bisects \( \angle BAC \). \( \overrightarrow{AD} \perp BC \)

Remark: \( \triangle ADB \equiv \triangle ADC \). Why?

Corollary (2)

The bisector of the vertex angle of an isosceles triangle bisects the base and is perpendicular to it.

In the figure opposite:

In \( \triangle ABC \), \( AB = AC \),

\( \overrightarrow{AD} \) bisects \( \angle BAC \)

then \( D \) is a midpoint of \( BC \) and \( \overrightarrow{AD} \perp BC \)

Remark: \( \triangle ADB \equiv \triangle ADC \). Why?
Corollary (3)

The straight line drawn passing through the vertex angle of an isosceles triangle perpendicular to the base bisects each of the base and the vertex angle.

In the figure opposite:
In \( \triangle ABC \), \( AB = AC \), \( \overline{AD} \perp \overline{BC} \)
then \( D \) bisects \( \overline{BC} \), \( m(\angle BAD) = m(\angle CAD) \)

Remark: \( \triangle ADB \cong \triangle ADC \). why?

Think

In the figure opposite:
\( ABCD \) is a quadrilateral in which all sides are equal in length, this figure is called rhombus, its diagonals are \( \overline{AC} \) and \( \overline{BD} \). they intersect at point \( M \)

Remark: \( \triangle ABD \cong \triangle CDB \). why?

\[ \therefore m(\angle ABD) = m(\angle CDB) \]
\[ \text{in } \triangle ABC, \ AB = BC, \ \overline{BM} \text{ bisects } \angle ABC \]
\[ \therefore BM \perp \ldots , \ M \text{ is the midpoint of } \overline{AC} \]
\[ \text{in } \triangle BAD, \ AB = AD, \ \overline{AM} \perp \overline{BD} \]
\[ \therefore AM \text{ bisects } \ldots , \ M \text{ is the midpoint of } \overline{BD} \]

Are the two diagonals of the rhombus perpendicular?
Do the two diagonals of the rhombus bisect each other?
Does the diagonal of the rhombus bisect the vertex angles which it connects?
Write down your answer.
First: axes of symmetry in the isosceles triangle:
The axis of symmetry of the isosceles triangle is the straight line drawn from the vertex angle perpendicular to its base.

In the figure opposite:
\( \triangle ABC \) in which \( AB = AC \), \( AD \perp BC \)
then \( AD \) is the axis of symmetry in the isosceles triangle \( ABC \).

Discuss:
Does the isosceles triangle have more than one axis of symmetry?
- How many axes of symmetry are there in the equilateral triangle?
- Are there any axes of symmetry in the scalene triangle?

Second: Axis of symmetry of a line segment:
The straight line perpendicular to a line segment at its middle is called the axis of symmetry for that line segment in brief it is known as the axis of a line segment.

In the figure opposite:
If \( D \) the midpoint of \( AB \) and 
The straight line \( L \perp AB \)
Where \( D \in L \), then the straight line \( L \)
Is the axis of \( AB \)

Important property
Any point at the axis of symmetry of a line segment is at equal distances from its end points.

Remark:
1. If \( C \in L \) then \( AC = BC \)
2. If \( EA = EB \) then \( E \in L \). why?
Examples

1. In the figure opposite

\[ AB = AC = 10 \text{ cm}, \quad EB = EC \]
\[ AE \cap BC = \{D\} \]
If \( BC = 6 \text{ cm}, \) find the length of \( CD \) and \( AD \)

**Given:** \( AB = AC, \quad EB = EC \)

**R.T.P.:** Find \( CD \) and \( AD \)

**Proof:**
\[ \therefore AB = AC \quad \therefore A \text{ is on the axis of } BC \]
\[ \therefore EB = EC \quad \therefore E \text{ is on the axis of } BC \]
\[ \therefore AE \text{ is the axis of } BC \]
\[ D \text{ is the midpoint of } BC, \quad AD \perp BC \]
\[ D \text{ is the midpoint of } BC, \quad BC = 6\text{ cm} \quad \therefore CD = 3\text{ cm} \]
\[ \therefore AD \perp BC \]
\[ \therefore \text{In } \Delta ADC \text{ that is right angled triangle at } D \]
\[ (AD)^2 = (AC)^2 - (CD)^2 \]
\[ (AD)^2 = 100 - 9 \]
\[ \therefore AD = \sqrt{91} \text{ cm} \]

2. In the figure opposite

\[ \text{A B C is a triangle in which } AB = AC, \quad AD \perp BC, \quad \angle B A D = 25^\circ, \]
\[ BC = 4 \text{ cm}. \text{ Find:} \]
\[ \hat{A} \quad \hat{m}(\angle D A C) \quad \hat{B} \quad \text{the length of } DC \]

**Solution**

**Given:** \( AB = AC, \quad AD \perp BC, \quad \angle BAD = 25^\circ, \quad BC = 4 \text{ cm} \)

**R.T.P.:** \( \hat{m}(\angle D A C), \) and the length of \( DC \).
Proof: In $\triangle ABC$
\[ \therefore AB = AC, \quad AD \perp BC \]
\[ \therefore AD \text{ bisects both the base } BC \text{ and } \angle BAC \]
\[ \therefore m(\angle DAC) = m(\angle DAB) = 25^\circ, \]
\[ DC = \frac{1}{2} BC = \frac{4}{2} = 2 \text{ cm}. \]

practice

1. In the figure opposite:
   \[ xy = xL, \quad Zy = ZL, \quad LM = YM \]

2. Prove that $X, M$ and $Z$ are on the same straight line

3. In the figure opposite:
   \[ BD = CE \]
   \[ m(\angle ABC) = m(\angle ACB) \]
   \[ m(\angle D) = m(\angle E) = 90^\circ \]
   Prove that: $m(\angle DAB) = m(\angle CAE)$

4. In the figure opposite:
   \[ AB = AC, \quad DE \parallel AB \]
   \[ DF \parallel AC \]
   Prove that: $DE = DF$

   Second: $m(\angle BAC) = m(\angle EDF)$
**Exercises (4-4)**

1. **Complete** to get a correct statement:
   - A. The vertex angle bisector in the isosceles triangle bisects the base and is ........
   - B. The number of symmetrical axes in the equilateral triangle is ........
   - C. Any point at the axis of a line segment symmetry is at two equal distances from ........
   - D. If the measurement of an angle in the isosceles triangle is 100°, then the measurement of an angle of the other two = ........°.

2. **Choose the correct answer:**
   - A. The number of axes of symmetry in the isosceles triangle = ........
     (0, 1, 2, 3)
   - B. The triangle whose sides lengths are 2 cm, (X + 3) and 5 cm becomes an isosceles triangle when X = ........ cm.
     (1, 2, 3, 4)
   - C. The intersecting point of the medians of a triangle divides each other from the direction of the base in a ratio ........
     (1:2, 2:1, 1:3, 2:3)

3. **In the figure opposite:**
   - AB = BC, BE bisects ∠CBD, D ∈ AB
   - Prove that: BE // AC

4. **In the figure opposite:**
   - \( \overline{AC} \cap \overline{BD} = \{M\} \)
   - \( \overline{AD} // \overline{BC} \), MB = MC
   - Prove that:
     1. \( \triangle AMD \) is an isosceles triangle
     2. The axis of symmetry of \( \triangle AMD \) is the same of \( \triangle BMC \).
1. In the figure opposite
   \( AB = AC, \ BC = 10\text{cm}, \)
   \( m(\angle BAD) = 30^\circ, \ AD \perp BC \)
   **First:** Find the length of \( BD, \ AD \).
   **Second:** How many axes of symmetry are there at \( \triangle ABC \)?
   **Third:** What is the area of \( \triangle ABC \)?

2. In the figure opposite
   \( AB = AC, \ DE \in AB, \ E \in AC \)
   \( BF \) bisects \( \angle DBC, \)
   \( CF \) bisects \( \angle BCE \)
   **Prove that**
   **First:** \( \triangle BFC \) is an isosceles triangle
   **Second:** \( AF \) is the axis of symmetry of \( BC \)

3. In the figure opposite
   \( AB = CB, \ AD = CD, \ E \in BD \)
   **Prove that**
   \( BD \) bisects \( \angle ADC \)
   \( DE \) bisects \( \angle ABC \)

4. In the figure opposite
   \( DE \parallel BC, \ AD = AE \)
   prove that: \( AB = AC \).
5. In the figure opposite
   AB = AC = AD = CD
   \( m(\angle BAC) = 40 \)
   **Find:** \( m(\angle BCD) \)

6. In the figure opposite
   ABC is a triangle which \( m(\angle B) = m(\angle C) \)
   **Find:** The perimeter of in the triangle

7. In the figure opposite
   ABCD is a quadrilateral in which \( \overline{AD} // \overline{BC} \), \( \overline{BD} \) bisects
   \( \angle ABC \), \( \overline{AE} \) bisects \( \angle BAD \)
   **Prove that**
   - \( AB = AD \)
   - \( \overline{AE} \perp \overline{BD} \)
   - \( BE = ED \)
Activity

1. Using a ruler and a compass, draw the acute-angle ABC. At the opposite side from \( \overrightarrow{BA} \). Draw \( \overrightarrow{AE} \parallel \overrightarrow{BC} \).

2. In the opposite figure \( ABCD \) is a rectangle in which \( \overrightarrow{AC} \) is a diagonal, \( \overrightarrow{AE} \) bisects \( \angle BAC \), \( \overrightarrow{DE} \perp \overrightarrow{AC} \), where \( \overrightarrow{AE} \cap \overrightarrow{DE} = \{E\} \), \( \overrightarrow{AC} \cap \overrightarrow{DE} = \{M\} \).

Prove that \( DA = DE \).
1. **Complete** the following to get a correct statement:

   A. The base angles in an isosceles triangle are ________________

   B. The median that is drawn from the vertex of an isosceles triangle is ________________ and ________________

   C. In the triangle ABC, AB = AC, m(∠A) = 70°, so m(∠C) = ________________

   D. The number of the axes of symmetry in an equilateral triangle = ________________

   E. The measure of the exterior angle in an equilateral triangle = ________________

   F. The straight line perpendicular to the midpoint of a line segment is called ________________

2. **In the figure opposite:**

   ABC is a triangle in which $\overrightarrow{BD}$ bisects $\angle ABC$ and intersects $\overrightarrow{AC}$ at D, $\overrightarrow{DE} \perp \overrightarrow{CB}$

   $\overrightarrow{DE} \cap \overrightarrow{AB} = \{E\}$

   **Prove that** $BE = ED$

3. **In the figure opposite** ABC is a triangle in which

   $\overrightarrow{AB} = \overrightarrow{AC}$, $\overrightarrow{AE}$ bisects $\angle BAC$,

   $\overrightarrow{AE} \cap \overrightarrow{BC} = \{E\}$

   $D \in \overrightarrow{AE}$.

   **Prove that**

   A. $BE = \frac{1}{2} BC$

   B. $BD = CD$
Think and Discuss

You will learn how

- To define the concept of inequality.
- To define axioms of inequality.

Key terms

- Inequality.
- Axioms.
- Greater than >.
- Less than <.
- Equal to =

The concept of inequality:

1. Do all the students in your class have the same height?
2. Are there any differences among the measures of acute, right, and obtuse angles?

What does this difference mean?

Remark:

An Inequality means that there is a difference in the heights of the students and in the measures of the angles. This difference is represented by the relation of inequality which is used to compare two different numbers.

Examples

1. If: $\angle ABC$ is an acute angle then: $m(\angle ABC) < 90^\circ$.
2. In the figure opposite, $ABC$ is a triangle in which:
   
   $AB = 4\text{cm}$, $BC = 3.5\text{cm}$,
   
   $AC = 2.4\text{cm}$

   then: $AB > BC$, $BC > AC$

   i.e $AB > BC > AC$
In the figure opposite, find: \( m(\angle ACB) \), \( m(\angle ACD) \)
and \( m(\angle ADE) \) then complete by using \( > \) or \( < \):

\[
\begin{align*}
m(\angle ADE) & \ldots \ldots m(\angle CAD) \\
m(\angle ADC) & \ldots \ldots m(\angle ACB) \\
m(\angle ACD) & \ldots \ldots m(\angle ABC) \\
m(\angle ACD) & \ldots \ldots m(\angle ADE)
\end{align*}
\]

Remark: All the previous relations are called inequalities.

Axioms of Inequality

For any given three numbers \( x, y \) and \( z \):

1. If: \( x > y \)  
   then: \( x + z > y + z \)

2. If: \( x > y \)  
   then: \( x - z > y - z \)

3. If: \( x > y \), \( z \) is a positive number  
   then: \( xz > yz \)

4. If: \( x > y \), \( y > z \)  
   then: \( x > z \).

5. If: \( x > y \), \( A > B \)  
   then: \( x + A > y + B \)
**Remember:** The measure of any exterior angle of a triangle is greater than the measure of any interior angle except for the adjacent angle.

**Practice**

1. In the figure opposite: which of the following angles has the greatest measure?
   - A \(\angle 1\), \(\angle 3\), \(\angle 4\)
   - B \(\angle 4\), \(\angle 8\), \(\angle 9\)
   - C \(\angle 2\), \(\angle 3\), \(\angle 7\)
   - D \(\angle 7\), \(\angle 8\), \(\angle 10\)

2. In the figure opposite, find:
   - A All angles of measures less than \(m(\angle 1)\)
   - B All angles of measures greater than \(m(\angle 6)\)
   - C All angles of measures less than \(m(\angle 4)\)

3. Order the measures of the angles in the triangle \(ABC\) in an ascending order and the measures of the angles in the triangle \(XYZ\) in a descending order.

4. In the figure opposite: \(C \in AB\), \(D \in AB\)
   - If: \(AB > CD\)
   - then: \(AC \ldots BD\)
Example

In the figure opposite:
\[ m(\angle ACD) > m(\angle ABD), \quad DB = DC \]

Prove that: \[ m(\angle ACD) > m(\angle ABD) \]

Given: \[ m(\angle ACD) > m(\angle ABD), \quad DB = DC \]

Required to prove: \[ m(\angle ACD) > m(\angle ABD) \]

R.T.P: \[ DB = DC \]

\[ \therefore m(\angle DBC) = m(\angle DCB) \quad (1) \]

\[ \therefore m(\angle ACD) > m(\angle ABD) \quad (2) \]

\[ \therefore \text{By subtracting } (1) \text{ from } (2), \text{ we get:} \]

\[ m(\angle ACD) - m(\angle DBC) > m(\angle ACD) - m(\angle DCB) \]

\[ \therefore m(\angle ACD) > m(\angle ABD) \quad Q.E.D \]

Exercises (5-1)

1. In the figure opposite:
   \[ \text{ABC is a triangle in which } AC > AB, \quad x \in \overrightarrow{AB} \]
   \[ y \in \overrightarrow{AC}, \text{where } m(\angle AXY) = m(\angle AYX) \]
   Prove that: \[ YC > XB \]

2. In the figure opposite:
   \[ \overrightarrow{AB} \parallel \overrightarrow{CD}, \quad AD \cap CB = \{M\}, \quad E \in \overrightarrow{CD}, \quad E \not\in \overrightarrow{CD} \]
   Prove that:
   \[ A: \quad m(\angle ACD) > m(\angle ABD) \]
   \[ B: \quad m(\angle ADE) > m(\angle ABD) \]

3. M is a point in a triangle ABC,
   Prove that: \[ m(\angle AMB) > m(\angle ACB) \]
UNIT FIVE
Lesson Two
Comparing the measures of the angles of a triangle

Think and Discuss

You will learn how
- To compare the measures of angles in a triangle.

Key terms
- Angle.
- Measure of an angle.
- The greatest angle in a triangle.
- The smallest angle in a triangle.
- The largest side of a triangle.
- The smallest side of a triangle.

Activity

1. In the figure opposite: ABC is an isosceles triangle in which AB = AC
   - Fold the triangle to make the vertex B congruent to vertex C. What do you observe regarding to the measures of the angles B, C which are opposite to the two equal sides AC, AB?
   - Fold the triangle to make the vertices A, C congruent. What do you observe regarding to the measures of the two angles opposite to the two unequal sides BC, AB?
   - Does the difference in the lengths of the two sides in a triangle lead to a difference in the measures of their two opposite angles?

2. Draw the scalene triangle. ABC Flip the triangle to make the vertex A coincide the vertex B. What do you observe regarding to the measures of the two angles A, and B that are opposite to the two unequal sides, BC, AC.
   - Repeat the previous steps to make the vertex B coincide vertex C. What do you observe?

First Term
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Are there any equal angles in measures in that triangle?

Notice that: In a triangle, if the sides are unequal in length, the measures of the opposite angles are unequal.

**Activity**

Draw the scalene triangle ABC, then measure the lengths of its 3 sides and the measures of the opposite angles, then complete the following table:

<table>
<thead>
<tr>
<th>Lengths of sides</th>
<th>Measures of the opposite angles</th>
</tr>
</thead>
<tbody>
<tr>
<td>AB = ........ cm</td>
<td>m ( ( \hat{C} ) ) = ........°</td>
</tr>
<tr>
<td>BC = ........ cm</td>
<td>m ( ( \hat{A} ) ) = ........°</td>
</tr>
<tr>
<td>CA = ........ cm</td>
<td>m ( ( \hat{B} ) ) = ........°</td>
</tr>
</tbody>
</table>

What do you observe?

**Theorem (3)**

(Angle - Comparison Theorem)

In a triangle, if two sides have unequal lengths, then the longer is opposite to the angle of the greater measure.

**Given:** ABC in which \( AB > AC \)

**R.T.P:** \( m ( \hat{ACB} ) > m ( \hat{ABC} ) \)

**Construction:** take \( D \in AB \) where \( AD = AC \)

**proof:** in \( \triangle ACD \), \( AD = AC \)

\[ m ( \hat{ACD} ) = m ( \hat{ADC} ) \quad (1) \]

\( ACD \) is an exterior angle of \( \triangle BDC \)

\[ m ( \hat{ADC} ) > m ( \hat{B} ) \quad (2) \]

From (1), (2)

\[ m ( \hat{ACD} ) > m ( \hat{B} ) \]

\[ m ( \hat{ACB} ) > m ( \hat{ACD} ) \]

\[ m ( \hat{ACB} ) > m ( \hat{ABC} ) \quad Q.E.D \]
Practice

In each of the following figures, complete using (> , < )

1

\[
\begin{array}{c|c|c}
\angle A & \angle B \\
m(\angle A) & m(\angle B) \\
\angle C & \angle D \\
m(\angle C) & m(\angle D)
\end{array}
\]

2

\[
\begin{array}{c|c|c}
\angle X & \angle Y \\
m(\angle X) & m(\angle Y) \\
\angle Z & \angle W \\
m(\angle Z) & m(\angle W)
\end{array}
\]

3

\[
\begin{array}{c|c|c}
\angle B & \angle C \\
m(\angle B) & m(\angle C) \\
\angle D & \angle E \\
m(\angle D) & m(\angle E)
\end{array}
\]

Remark:

The measure of the greatest angle in the triangle > 60°

The measure of the smallest angle in the triangle is < 60° why?

Example

In the figure opposite:

ABC is a triangle in which AB > BC > CA

Prove that: m(\angle C) > m(\angle A) > m(\angle B)

Given: AB > BC > CA

R.T.P: m(\angle C) > m(\angle A) > m(\angle B)

Proof:

In \(\triangle ABC\)

1. \(AB > BC\) \(\therefore m(\angle C) > m(\angle A)\) (1)

2. \(BC > CA\) \(\therefore m(\angle A) > m(\angle B)\) (2)

From (1), (2) and using the axioms of inequality:

\(m(\angle C) > m(\angle A) > m(\angle B)\)
Remember: In a triangle, the longest side in length is opposite to the greatest angle in measure while the shortest side in length is opposite to the smallest angle in measure.

Example

In the figure opposite:

ABC is a triangle where $\overrightarrow{BM}$ bisects $\angle ABC$, and $\overrightarrow{CM}$ bisects $\angle ACB$ if: $MC > MB$

Prove that: $m(\angle ABC) > m(\angle ACB)$

Given: $\overrightarrow{BM}$ bisects $\angle ABC$, $\overrightarrow{CM}$ bisects $\angle ACB$, $MC > MB$.

R.T.P: Prove that $(\angle ABC) > m(\angle ACB)$

Proof: In $\triangle MBC$

$\therefore MC > MB \quad \therefore m(\angle MBC) > m(\angle MCB)$ (1)

In $\triangle ABC$

$\therefore \overrightarrow{BM}$ bisects $\angle ABC \quad \therefore m(\angle MBC) = \frac{1}{2} m(\angle ABC)$ (2)

$\therefore \overrightarrow{CM}$ bisects $\angle ACB \quad \therefore m(\angle MCB) = \frac{1}{2} m(\angle ACB)$ (3)

$\therefore$ from (1), (2), (3): $\frac{1}{2} m(\angle ABC) > \frac{1}{2} m(\angle ACB)$ Using the axioms of inequality

$\therefore m(\angle ABC) > m(\angle ACB)$ Q.E.D

Exercise (5-2)

1. $\triangle ABC$ in which $AB = 2.7cm$, $BC = 8.5cm$, $AC = 6cm$. Order the measures of the angles of the triangle ascendingly.

2. In the figure opposite $XY > XL$, $YZ > ZL$

Prove that: $m(\angle XLZ) > m(\angle XYZ)$
3. In the figure opposite:

\[ BM \] is a median in the triangle \( ABC \), \( BM < AM \)

Prove that: \( \angle ABC \) is an obtuse angle.

4. In the figure opposite:

\( ABC \) is a triangle, \( AB > AC \)

\( \overrightarrow{XY} \parallel \overrightarrow{BC} \)

Prove that: \( m(\angle AXY) > (\angle AXY) \)

5. In the figure opposite:

\( A \ B \ C \) is a triangle, \( B M \) bisects

\( \angle A B C \),
\( CM \) bisects \( \angle ACB \). If \( AB > AC \)

Prove that: \( m(\angle MCB) > m(\angle MBC) \)

6. In the figure opposite:

\( ABCD \) is a quadrilateral in which \( AD = DC \),
\( BC > AB \)

Prove that:

\( m(\angle A) > m(\angle C) \)

7. \( ABCD \) is a quadrilateral in which \( AB \) is the longest side in length, \( CD \) is the shortest one. Prove that \( m(\angle BCD) > m(\angle BAD) \)
Comparing the lengths of sides of a triangle

Think and Discuss

Activity 1 The figure opposite: ABC is a triangle of unequal measures of angles.

Fold the triangle to make the vertex A coincide vertex B what do you observe regarding to the lengths of the two sides BC and AC, which are opposite to the two unequal angles A and B?

Repeat the same previous steps to make vertex B congruent to vertex C. What do you observe?

When vertex C is coincide to vertex A, what do you observe?

Are there any equal sides in lengths in that triangle?

Remark: If the measures of the angles in a triangle are unequal, then the lengths of its sides which are opposite to the angles are unequal.

Activity 2 Draw the triangle ABC where its angles are unequal in measure then measure the lengths of opposite sides to the angles and complete the following table:

<table>
<thead>
<tr>
<th>the measures of the angles</th>
<th>the lengths of the opposite sides</th>
</tr>
</thead>
<tbody>
<tr>
<td>m (∠ A) = ....*</td>
<td>BC = .... cm</td>
</tr>
<tr>
<td>m (∠ B) = ....*</td>
<td>CA = .... cm</td>
</tr>
<tr>
<td>m (∠ C) = ....*</td>
<td>AB = .... cm</td>
</tr>
</tbody>
</table>

What do you observe?

Is the greatest angle in measure opposite to the longest side in length? Is the smallest angle in measure opposite to the shortest side in length?

Is it possible to order the lengths of the sides in the triangle in an ascending or descending order in terms of the measures of the opposite angles?

You will learn how

To compare the measures of sides in a triangle.

Key terms

- The longest side of a triangle.
- The shortest side of a triangle.
- The greatest angle of a triangle.
- The smallest angle of a triangle.
- The perpendicular line segment.
Theorem (4)

(Side - Comparison Theorem)

In a triangle, if two angles are unequal in measure, then the greater angle in measure is opposite to a side greater in length than that opposite to the other angle.

**Given:** \( \triangle ABC \) \( m(\angle C) > m(\angle B) \)

**R. T. P:** \( AB > AC \)

**Proof:** \( \therefore \) \( AB \), \( AC \) are line segments

\( \therefore \) one of the following cases should be verified:

1. \( AB < AC \)
2. \( AB = AC \)
3. \( AB > AC \)

If not \( AB > AC \)

Either \( AB = AC \) or \( AB < AC \)

if \( AB = AC \), then \( m(\angle C) = m(\angle B) \)

Again this contradicts the given where \( m(\angle C) > m(\angle B) \)

and if \( AB < AC \), then \( m(\angle C) < m(\angle B) \). According to the theorem above.

Again this contradicts the given, where

\( m(\angle C) > m(\angle B) \)

\( \therefore AB > AC \)

Q.E.D

**practice**

In the following figures, complete using >, < or =:

1.

\[ \angle A = 70^\circ, \angle B = 75^\circ \]

\( AB \) ...... \( AC \)

\( AB \) ...... \( BC \)

\( AC \) ...... \( BC \)

2.

\[ \angle X = 40^\circ, \angle Y = 30^\circ \]

\( XY \) ...... \( XZ \)

\( YZ \) ...... \( XY \)

\( YZ \) ...... \( XZ \)
Corollaries:

Corollary (1)

In the right-angled triangle, the hypotenuse is the longest side.

In the figure opposite \( \triangle ABC \) is a right-angled triangle at B.

\[ \therefore \angle A \text{ acute} \quad \therefore m(\angle B) > m(\angle A) \]
\[ AC > BC \]

\[ \therefore \angle C \text{ acute} \quad \therefore m(\angle B) > m(\angle C) \]
\[ AC > AB \]

\[ \therefore AC \text{ is the longest side} \quad \text{Q.E.D.} \]

Remark: In the obtuse angled triangle, the side opposite to the obtuse angle is the longest side in the triangle.

Let's think

\[ AC > AB. \text{ Why?} \]
\[ AD > AB. \text{ Why?} \]
\[ AE > AB. \text{ Why?} \]

Is the length of the right leg in the right angled triangle is shorter than the length of the hypotenuse? Why?
Corollary (2)

The length of the perpendicular line segment drawn from a point outside a straight line to this line is shorter than any line segment drawn from this point to the given straight line.

Definition:
The distance between any point and a given straight line is the length of the perpendicular line segment drawn from the point to the given line.

Example

In the figure opposite: ABC is a triangle, \( \overrightarrow{AD} \parallel \overrightarrow{BC} \), \( m(\angle CDA) = 35^\circ \)
\( m(\angle DAE) = 75^\circ \)

Prove that: \( AC > AB \)

Given that: \( \overrightarrow{AD} \parallel \overrightarrow{BC} \), \( m(\angle EAD) = 75^\circ \), \( m(\angle DAC) = 35^\circ \)

R.T.P.: \( AC > AB \)

Proof:
\[ \therefore \overrightarrow{AD} \parallel \overrightarrow{BC} \]
\( AB \) is a transversal

\[ \therefore m(\angle B) = m(\angle EAD) = 75^\circ \]
\[ \therefore \overrightarrow{AD} \parallel \overrightarrow{BC} \]
\( AC \) is a transversal

\[ \therefore m(\angle ACB) = m(\angle DAC) = 35^\circ \]

From (1) and (2):

in \( \triangle ABC \)

\[ m(\angle ABC) = 75^\circ \]
\[ m(\angle ACB) = 35^\circ \]

i.e. \( m(\angle ACB) > m(\angle A \triangle B) \)

\[ \therefore AC > AB \]

Q.E.D
Exercise (5-3)

1. ΔABC in which m(∠A) = 40°, m(∠B) = 75°. Order the lengths of sides of the triangle descendingly.

2. In the figure opposite:
   \[ \overrightarrow{AD} \parallel \overline{BC}, \quad m(∠BAC) = 80° \]
   \[ m(∠DAC) = 30°. \] Prove that: BC > AB

3. In the figure opposite:
   In ΔABC is a triangle, D ∈ \overline{BC} where BD = AD
   Prove that: BC > AC

4. In the figure opposite:
   ABC is a triangle, \overrightarrow{CD} bisects ∠C and intersects \overrightarrow{AB} at point D.
   \[ m(∠BDC) = 100°, \quad DB = DC \]
   Prove that: AC > DB.

5. In the figure opposite:
   ABC is a triangle, D ∈ \overrightarrow{CB}, E ∈ \overrightarrow{AC}
   \[ m(∠ABD) = 110°, \quad m(∠BCE) = 120° \]
   Prove that: AB > BC.
6. In the figure opposite:
\[ AB \cap CD = \{M\}, \quad AC \perp CD, \quad BD \perp CD \]
prove that: \( AB > CD \)

7. In the figure opposite:
\[ \Delta ABC \text{ is an obtuse-angled triangle at B} \]
\[ DE \parallel BC \]
Prove that: \( AE > AD \)

8. \( \Delta ABC \) is a triangle, \( \overrightarrow{CD} \) bisects \( \angle C \), \( CD \cap AB = \{D\} \), prove that \( BC > BD \)

9. \( \Delta ABC \) in which \( m(\angle A) = (5x + 2)\), \( m(\angle B) = (6x - 10)\), \( m(\angle C) = (x + 20)\),
order the lengths of the sides of the triangle ascendingly.

10. In the figure opposite:
\[ \Delta ABC \text{ is a triangle, } D \in \overrightarrow{BC}, \quad AB = AD = BD \]
Prove that: \( BC > AC \)

11. \( \Delta ABC \) is a right-angled triangle at \( B \), \( D \in \overrightarrow{AC}, \quad E \in \overrightarrow{BC} \), where \( AD = BE \)
Prove that: \( m(\angle CED) > m(\angle CDE) \)
Think and Discuss

Activity

By using your ruler and compass, try to draw the triangle ABC where:

1. \( A\ B = 4 \text{ cm} \), \( B\ C = 5 \text{ cm} \), \( A\ C = 6 \text{ cm} \)
2. \( A\ B = 6 \text{ cm} \), \( B\ C = 3 \text{ cm} \), \( A\ C = 2 \text{ cm} \)
3. \( A\ B = 9 \text{ cm} \), \( B\ C = 4 \text{ cm} \), \( A\ C = 3 \text{ cm} \)
4. \( A\ B = 8 \text{ cm} \), \( B\ C = 3 \text{ cm} \), \( A\ C = 5 \text{ cm} \)

In which of the previous cases were you able to draw the triangle? What do you conclude?

Fact: For any triangle, the sum of the lengths of any two sides is greater than the length of the third side.

\[
\begin{align*}
\text{i.e.: } & \quad A\ B + B\ C > A\ C \\
& \quad B\ C + C\ A > A\ B \\
& \quad A\ B + A\ C > B\ C
\end{align*}
\]

For example: the numbers 5, 3 and 9 are not valid to be the lengths of a triangle because the sum of the smallest two numbers = 3 + 5 = 8, 8 < 9. Therefore, the inequality of the triangle is not verified.

Examples

ABC is a triangle, if \( A\ B = 10 \text{ cm} \), \( B\ C = 8.5 \text{ cm} \)

Find the interval which the length of side \( \frac{A\ C}{A\ C} \) belongs to.
Solution

\[ AC < AB + BC \quad \quad AC < 18.5 \quad (1) \]

However, \[ AC + BC > AB \]

\[ AC > AB - BC \quad \quad AC > 1.5 \quad (2) \]

From (1), (2) \[ 18.5 > AC > 1.5 \]

\[ AC \in [1.5, 18.5] \]

Practice

Find the interval which the third side belongs to in each of the following triangles. If the lengths of the other two sides were as follows:

- A 6 cm, 9 cm
- B 5 cm, 12 cm
- C 7 cm, 15 cm
- D 2.9 cm, 3.2 cm

Exercise (4-4)

1. If the lengths of two sides in an isosceles triangle were 5 cm, 12 cm, Find the length of the third side (State the reason).

2. Which of the following groups are valid in drawing a triangle?
   - A 5 cm, 7 cm, 8 cm
   - B 4 cm, 9 cm, 3 cm
   - C 10 cm, 6 cm, 4 cm
   - D 15 cm, 17 cm, 30 cm.

3. Prove that the length of any side in a triangle is less than half of the perimeter of the same triangle.

4. In the figure opposite:
   ABC is a triangle in which M is a point inside it. Prove that:
   \[ MA + MB + MC > \frac{1}{2} \text{ perimeter of the triangle ABC} \]

5. Prove that: the sum of the lengths of the two diagonals in a convex quadrilateral is less than its perimeter.
General Exercises

1. In the figure opposite: \( \triangle ABC \) is an equilateral triangle where \( E \) is a point inside it 
   \( m(\angle ECB) > m(\angle EBC) \).
   First: prove that: \( m(\angle ABE) > m(\angle ACE) \).
   Second: \( m(A) > m(\angle ABE) > m(\angle ACE) \).

2. In the figure opposite:
   \( DB = DC \).
   \( m(\angle ABC) > m(\angle ACB) \)
   Prove that: \( m(\angle ABD) > m(\angle ACD) \).

3. \( \triangle ABC \) is a triangle in which \( AB = 6\text{cm}, AC = 7\text{cm}, BC = 8\text{cm} \).
   Order the measures of its angles ascendingly.

4. In the figure opposite:
   \( AB > AC, DB = DC \)
   Prove that: \( m(\angle BAD) < m(\angle CAD) \).

5. In the figure opposite:
   \( XZ > XY \)
   \( XL \perp ZY \)
   prove that \( m(\angle LXZ) > m(\angle LXZ) \).
6. In the figure opposite:
ABCD is a quadrilateral in which
AB = AD = 5cm,
BC = 2 cm, DC = 4cm.
Prove that: \( \hat{m} (ABC) > \hat{m} (ADC) \)

7. In the figure opposite:
\( \hat{m} (DBC) = 140^\circ \)
\( \hat{m} (ECB) = 120^\circ \)
Prove that: CB > AB

8. In the figure opposite:
AB = AC
\( \hat{m} (ABC) = 65^\circ \)
\( \hat{m} (ACD) = 20^\circ \)
Prove that: AB > AD

9. In the figure opposite:
\( \hat{m} (B) = 90^\circ \)
Prove that: AC > DC

10. In the figure opposite:
AC > DC, \( \hat{m} (CAD) = \hat{m} (BAD) \)
AE = AC
Prove that:
- A DE = DC
- B \( \hat{m} (BAD) > \hat{m} (ADC) \)
- C BD > DC
1. In the given figure complete using (< or >)
   - A: $m(\angle D A C)$ ........ $m(\angle A C D)$
   - B: $m(\angle A E C)$ ........ $m(\angle E C A)$
   - C: $m(\angle A B E)$ ........ $m(\angle E A B)$
   - D: $m(\angle C D A)$ ........ $m(\angle D A C)$
   - E: $m(\angle A E B)$ ........ $m(\angle E A C)$

2. In the triangle $ABC$, $AB = 6\text{cm}$, $BC = 9\text{cm}$
   Then $AC \in \ldots$, \ldots[

3. In the triangle $ABC$: $m(\angle A) = (9x)^\circ$, $m(\angle B) = (6x - 17)^\circ$
   $m(\angle C) = (7x - 1)^\circ$
   Order the lengths of the sides of the triangle ascendingly.
Unit Test

1. Complete the following to make each statement true:
   A. In a triangle, the smallest angle in measure is opposite to .......
   B. In \( \triangle ABC \): if \( m(\angle A) = 70^\circ \), \( m(\angle B) = 30^\circ \), then the longest side in length is .......
   C. If the lengths of two sides in an isosceles triangle were: 3 cm, 7 cm then, the length of the third side = ....... cm
   D. \( \triangle ABC \) in which: \( m(\angle A) = 100^\circ \), then the greatest side length is .......
   E. \( \triangle ABC \) in which \( AB = 3 \text{ cm}, BC = 5 \text{ cm}, \text{ then } AC \in \) ......., .......
   F. The longest side length in the right-angled triangle is .......

2. In the figure opposite:
   ABCD is a quadrilateral in which
   \( AB = 6 \text{ cm}, BC = 4 \text{ cm}, \)
   \( CD = 7 \text{ cm}, DA = 8 \text{ cm} \)
   Prove that:
   \( m(\angle BCD) > m(\angle BAD) \)

3. In the figure opposite:
   \( \triangle ABC \) is a triangle, \( \overrightarrow{BD} \) bisects \( \angle B, \overrightarrow{BD} \cap \overrightarrow{AC} = \{D\}, \overrightarrow{DE} \parallel \overrightarrow{CB} \) and intersects \( \overrightarrow{AB} \) at \( E \)
   Prove that: \( AB > AD \)

4. In the figure opposite:
   \( \triangle ABC \) in which \( AB > AC, D \in \overrightarrow{AB}, E \in \overrightarrow{AC} \)
   \( \overrightarrow{BF} \) bisects \( \angle DBC, \overrightarrow{CF} \) bisects \( \angle BCE \)
   \( \overrightarrow{BF} \cap \overrightarrow{CF} = \{F\} \)
   Prove that:
   A. \( m(\angle FBC) > m(\angle BCF) \)
   B. \( CF > BF \)
Model tests on Algebra and statistics (2nd preparatory)

**Model (1)**

**First question:** complete the following:

1. The S.S of the equation \((x^2 + 3)(x^3 + 1) = 0\) is \(x \in \mathbb{R}\).
2. If the lowest boundary of a set is 10 and the upper boundary is \(x\) and its centre is 15, then \(x = \ldots\).
3. \([−2, 2] \cup \{−2, 0\} = \ldots\)
4. A cube whose volume is 8 cm³, then the sum of the lengths of all its edges equals \(\ldots\) cm.
5. The multiplicative inverse of the number \(\sqrt{3} + \sqrt{2}\) is \(\ldots\) (in the simplest form).

**Second question:** Choose the correct answer from those given.

1. If the radius of a sphere is 6 cm, then its volume is \(\ldots\)
   - a. \(6\pi\text{ cm}^3\)
   - b. \(36\pi\text{ cm}^3\)
   - c. \(72\pi\text{ cm}^3\)
   - d. \(288\pi\text{ cm}^3\)
2. If the point \((a,1)\) satisfies the relation \(x+y=5\), then \(a = \ldots\)
   - a. \(1\)
   - b. \(-4\)
   - c. \(4\)
   - d. \(5\)
3. \((2\sqrt{3})^3 = \ldots\)
   - a. \(4\)
   - b. \(8\)
   - c. \(16\)
   - d. \(40\)
4. The median of the values 34, 23, 25, 40, 22, 4 is \(\ldots\)
   - a. \(22\)
   - b. \(23\)
   - c. \(24\)
   - d. \(25\)
5. If the arithmetic mean of the values 27, 8, 16, 24, 6, \(k\) is 14 then \(k = \ldots\)
   - a. \(3\)
   - b. \(6\)
   - c. \(27\)
   - d. \(84\)
6. In the opposite figure
   the value of the mode = \(\ldots\)
   - a. \(4\)
   - b. \(5\)
   - c. \(6\)
   - d. \(40\)

**question (3)**

(a) Find the value of \(\sqrt{18} + \frac{3\sqrt{54} - 3\sqrt{2}}{2} - \frac{3}{2}\sqrt{16}\)

(b) If \(X = \frac{3}{\sqrt{5} - \sqrt{2}}\), \(Y = \sqrt{5} - \sqrt{2}\), prove that: \(X, Y\) are two conjugate

**question (4)**

(a) The area of a square is 1089 cm². Find the length of its diagonal.

(b) Find the S.S of the inequality \(\frac{3X + 1}{6} < X + 1 < \frac{X + 4}{2}\) in \(\mathbb{R}\), then represent it on the number line.
**Question (5)**

(a) The radius length of the base of a right circular cylinder, its base radius length is 4 \(4\sqrt{2}\) cm and its height is 9 cm find its volume in terms of \(\pi\) and if its volume equals the volume of a sphere find the radius length of the sphere.

(b) Find the arithmetic mean of the following frequency distribution.

<table>
<thead>
<tr>
<th>The set</th>
<th>5 –</th>
<th>15 –</th>
<th>25 –</th>
<th>35 –</th>
<th>45 –</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>7</td>
<td>10</td>
<td>12</td>
<td>13</td>
<td>8</td>
<td>50</td>
</tr>
</tbody>
</table>

---

**Question (2)**

Q. **Complete the following:**

(1) The additive inverse of the number \(-\sqrt{3} - \sqrt{5}\) is 

(2) \((\sqrt{8} + \sqrt{2}) (\sqrt{8} - \sqrt{2}) = \) 

(3) The conjugate of the number \(\frac{2\sqrt{5} - 3\sqrt{2}}{\sqrt{2}}\) is 

(4) If the volume a sphere is \(\frac{9}{2}\pi\) cm\(^3\) then its diameter length is \(\) cm.

(5) \(\{3,4\} - \{3,5\} = \) 

Q. **Choose the correct answer from those given:**

(1) If the volume of a cube is 27 cm\(^3\) then the area of one of its faces is 
  a. 3 cm\(^2\)  
  b. 9 cm\(^2\)  
  c. 36 cm\(^2\)  
  d. 54 cm\(^2\)

(2) If the mode of the set of values 4, 11, 8, 2 \(X\) is 4 then \(X = \) 
  a. 2  
  b. 4  
  c. 6  
  d. 8

(3) If the arithmetic mean of the set of values 18, 23, 29, 2k-1, k is 18 then \(k = \) 
  a. 1  
  b. 7  
  c. 29  
  d. 90

(4) If the lowest limit of a set is 4 and the upper limit is 8 then its centre is 
  a. 2  
  b. 4  
  c. 6  
  d. 8

(5) A right circular cylinder, the radius of its base is r, its height equals the length of the diameter, then its volume = 
  a. \(\pi r^3\)  
  b. \(\pi r^2\)  
  c. \(2\pi r^3\)  
  d. \(2r^3\)

(6) The solution set in R of the equation \(x(x^2 - 1) = 0\) is 
  a. \(\{0\}\)  
  b. \(\{1\}\)  
  c. \(-1\)  
  d. \(\{0, -1, 1\}\)
question (3)
(a) Reduce to the simplest form, \( \frac{\sqrt{3}}{\sqrt{5} - \sqrt{3}} + \frac{\sqrt{5}}{\sqrt{5} + \sqrt{3}} \)
(b) Prove that \( \sqrt{\frac{1}{128}} + \sqrt{\frac{1}{16}} - 2 \sqrt{\frac{3}{54}} = 0 \)

question (4)
(a) Find the S.S of the inequality \( -2 < 3x + 7 \leq 10 \) in \( \mathbb{R} \), then represent the interval of solution on the number line.
(b) If \( x = \sqrt{2} + \sqrt{3} \) find the value of \( x^4 - 2x^2 + 1 \)

question (5)
(a) The opposite figure represents the ascending cumulative curve of the marks 32 students in one test complete the median mark = .................
(b) Find the arithmetic mean of the following frequency distribution.

<table>
<thead>
<tr>
<th>The set</th>
<th>5 –</th>
<th>15 –</th>
<th>25 –</th>
<th>35 –</th>
<th>45 –</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>3</td>
<td>2</td>
<td>20</td>
</tr>
</tbody>
</table>

Model Test (3)

"Merge students"

Questions (1) Complete each of the following:
1) The conjugate of the number \( \sqrt{3} + \sqrt{2} \) is ......................
2) \( \sqrt{18} + \sqrt{54} - 3\sqrt{2} = ......................... \)
3) The mode for the number 3, 5, 3, 4, 3 is ......................
4) The median of the values 2, 3, 5, 7, 9 is ......................
5) The solution set of the equation \( x^2 + 9 = 0 \) in \( \mathbb{R} \) is ......................

Questions (2) Choose correct answer:
1) The arithmetic mean for the values 9, 6, 5, 14, 1 is ......................
   (a) 7 (b) 3 (c) 5 (d) 9
2) The simplest form of the expression \( (\sqrt{3} - \sqrt{2}) (\sqrt{3} + \sqrt{2}) \) is ......................
   (a) \( \sqrt{3} \) (b) 1 (c) \( \sqrt{2} \) (d) \( 2 \sqrt{3} \)
3) The additive inverse of the number - \( \sqrt{5} \) is ......................
   (a) \( \sqrt{5} \) (b) 5 (c) \( \sqrt{2} \) (d) -5
4) \( [3, 5] - (3, 5) = \) ………………

(a) \( [3, 5] \)  
(b) \([3, 5]\)  
(c) \( \phi \)  
(d) \( [3, 5] \)

5) A cube of volume 64 cm\(^3\), then the edge length is ………… cm.

(a) 4  
(b) 8  
(c) 16  
(d) 64

Questions (3) **Match from the column (A) to the suitable one from the column (B):**

<table>
<thead>
<tr>
<th>(A)</th>
<th>(B)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) The solution set of the equation ( x^2 - 25 = 0 ) R is</td>
<td>([0, 2])</td>
</tr>
<tr>
<td>2) ([-3, 2] \cap [0, 2]) = …………..</td>
<td>7</td>
</tr>
<tr>
<td>3) If the order of the median is the fourth, then the number of the values is …………..</td>
<td>({5, -5})</td>
</tr>
<tr>
<td>4) (\sqrt{3}) is an ………….. number.</td>
<td>(\overline{3} \overline{7})</td>
</tr>
<tr>
<td>5) The solution set of the inequality (3 \leq x \leq 7) on the number line is …………..</td>
<td>Irrational</td>
</tr>
</tbody>
</table>

Questions (4) **Put (✓) for the correct statement, (x) for the incorrect one:**

1) The arithmetic mean for a set of values = the sum of these values +
   its number.

2) If \(x = \sqrt{13} - \sqrt{7}, y = \sqrt{13} + \sqrt{7}\), then \(x, y\) are two conjugate numbers.

3) The irrational number \(\sqrt{7}\) lies between 2 and 3

4) \(\sqrt{75} - 2\sqrt{27} = 7\sqrt{3}\)

5) The simplest from of the number \(\frac{1}{\sqrt{5}}\) is \(\frac{\sqrt{5}}{5}\)

Questions (5) **First: Complete:**

If the lower limit of a set is 4, the upper limit is 8, then its centre = \(\frac{4 + 8}{2}\) = …………..

First Term
**Second : Complete**

The following table to obtain the arithmetic mean of the following frequency distribution:

<table>
<thead>
<tr>
<th>Sets</th>
<th>5-</th>
<th>15-</th>
<th>25-</th>
<th>35-</th>
<th>45-</th>
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<td>50</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Sets</th>
<th>The centre of the set x</th>
<th>Frequency (F)</th>
<th>F x X</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 –</td>
<td>10</td>
<td>7</td>
<td>10 x 7 = 70</td>
</tr>
<tr>
<td>15 –</td>
<td>20</td>
<td>10</td>
<td>20 x 10 = ……………</td>
</tr>
<tr>
<td>25 –</td>
<td>……………</td>
<td>……………</td>
<td>…………… x 12 = ………</td>
</tr>
<tr>
<td>35 –</td>
<td>……………</td>
<td>……………</td>
<td>…………… x 13 = ………</td>
</tr>
<tr>
<td>45 –</td>
<td>……………</td>
<td>……………</td>
<td>…………… x 8 = ………</td>
</tr>
<tr>
<td>The sum</td>
<td>……………</td>
<td>50</td>
<td></td>
</tr>
</tbody>
</table>

The arithmetic mean = $\frac{\sum(F \times X)}{\sum(F)} = \frac{\ldots}{\ldots} = \ldots$
1. **Complete the following:**

   The longest side in the right angled triangle is .................
   
   If the lengths of two sides in a triangle are 2 cm. and 7 cm. then ............... the length of the third side < .................
   
   If the measures of two angles in a triangle are different, then the greater in measure of them is opposite to .................
   
   If the median drawn from a vertex of a triangle equals half the opposite side to this vertex in length then .................
   
   If the measure of an angle in the isosceles triangle equals 60° then the triangle is .................

2. **Choose the correct answer from those given:**

   In the opposite figure:
   
   $\triangle ABC$ is equilateral, then $m\ (\angle ACD) =$ .................
   
   (a) $45^\circ$  (b) $60^\circ$  (c) $120^\circ$  (d) $135^\circ$
   
   In $\triangle ABC$ which is right angled at $B$, if $AC = 20$ cm. then the length of the median of the triangle drawn from $B$ equals .................
   
   (a) $10$ cm.  (b) $8$ cm.  (c) $6$ cm.  (d) $5$ cm.

   $\triangle XYZ$ is a triangle in which $m\ (\angle Z) = 70^\circ$ and $m\ (\angle Y) = 60^\circ$ then $YZ$ .................
   
   XY...... (a) $>$  (b) $<$  (c) $=$  (d) twice

   The lengths which can be lengths of sides of a triangle are
   
   (a) $(0, 3, 5)$  (b) $(3, 6, 5)$  (c) $(3, 3, 6)$  (d) $(3, 3, 7)$

   The triangle in which the measure of two angles of it are $42^\circ$ and $69^\circ$ is .................
   
   (a) an isosceles triangle.  (b) an equilateral triangle.
   
   (c) a scalene triangle.  (d) a right angled triangle.

   In the opposite figure, $m\ (\angle A) = 2m\ (\angle C)$, $AC =$ ................. cm.
   
   (a) $3$  (b) $6$  (c) $9$  (d) $12$

3. **Complete :**

   [a] If $\triangle ABC$ in which $AB > AC$ then $m\ (\angle C) =$ ................. $m\ (\angle B)$

   [b] **In the opposite figure :**
   
   $m\ (\angle A) = 50^\circ$, $AB = AC$
   
   and $\triangle DBC$ is an equilateral triangle

   Find $m\ (\angle ABD)$

   [c] **In the opposite figure :**
   
   $AD \parallel BC$, $m\ (\angle BAC) = 70^\circ$

   and $m\ (\angle DAC) = 50^\circ$

   Prove that : $BC > AC$
4. [a] Prove that the base angles of the isosceles triangle are congruent.

[b] In the opposite figure:
\[ AB = AC, BD \text{ bisects } \angle B \text{ and } CD \text{ bisects } \angle C \]
Prove that: \( \triangle DBC \) is isosceles.

5. [a] In the opposite figure:
Arrange the angles of \( \triangle ABC \) descendingly due to their measures.

[b] In the opposite figure:
\[ AB > BC, \overline{XY} \parallel \overline{BC} \]
Prove that:
\[ AX > XY \]

Model (2)

1. Choose the correct answer from those given:
   1. The triangle which has three axes of symmetry is ..............
      (a) scalene   (b) isosceles   (c) right angled  (d) equilateral.
   2. The sum of lengths of two sides in a triangle is .............. the length of the third side.
      (a) greater than (b) smaller than (c) equals to (d) twice
   3. If the lengths of two sides in an isosceles triangle are 8 cm. and 4 cm. then the length of the third side is ..............
      (a) 4 (b) 8 (c) 3 (d) 12
   4. In \( \triangle ABC \) if \( \angle B = 130^\circ \) then the tallest side of it is ..............
      (a) \( BC \) (b) \( AC \) (c) \( AB \) (d) its median
   5. \( \triangle XYZ \) is an isosceles triangle in which \( \angle X = 100^\circ \) then \( \angle Y = \) ..............
      (a) 100 (b) 80 (c) 60 (d) 40
   6. In the opposite figure \( x+y = \) ..............
      (a) 100 (b) 140 (c) 180 (d) 280

2. Complete the following:
   1. If the measure of an angle in a right angled triangle is \( 45^\circ \) then the triangle is ..............
   2. The length of any side in a triangle .............. the sum of lengths of the two other sides.
   3. If \( \overline{AB} = \overline{XY} \) then \( AB = \) ..............
   4. In \( \triangle ABC \) of \( \angle A = 30^\circ \) and \( \angle B = 90^\circ \) then \( BC = \) .............. \( AC \)
   5. The axis of symmetry of a line segment is the straight line which .............. at its midpoint.

3. [a] In \( \triangle ABC \) in which \( AB = 7 \text{ cm}, \ BC = 5 \text{ cm. and } AC = 6 \text{ cm.} \) Arrange its angles ascendingly due to their measure.

[b] In the opposite figure:
\( \triangle ABC \) is right angled at \( B \),
m(\( \angle C \)) = 30° \( \Rightarrow \) \( D \) is the midpoint of \( \overline{AC} \)
\( \Rightarrow \) \( E \) is the midpoint of \( \overline{BC} \), \( AC = 9 \text{ cm.} \)
Find the length of each of \( BD, BM \) and \( AB \)
4. **[a] In the opposite figure:**
   \( m(\angle ABC) = m(\angle BDE) = 90^\circ \)
   \( m(\angle E) = 30^\circ \)
   D is the midpoint of \(\overline{AC}\)
   Prove that: \(AC = BE\)

   **[b] In the opposite figure:**
   \(\overline{AD} \parallel \overline{BC}\)
   \( m(\angle BAC) = 70^\circ \)
   \( m(\angle DAC) = 30^\circ \)
   Prove that:
   \(AC > BC\)

5. **[a] If the measures of two angles of a triangle are different, then their greater in measure is opposite to ...............**

   **[b] In the opposite figure:**
   \(\overline{AB} \parallel \overline{XY}\) and \(AB\) bisects \(\angle YAZ\)
   Prove that:
   \(XZ > YZ\)

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**Model Test**

"Merge students"

**Questions (1) Complete:**

1. The point of concurrence of the medians of the triangles divides each median in the ratio of .............. : .............. from its ......................

2. In the right-angled triangle, the length of the median drawn from of the right angle equals ......................

3. The base angles of the isosceles triangle are ....................

4. In \(\triangle ABC\), \(m(\angle B) = 70^\circ\), \(m(\angle C) = 50^\circ\), then \(AC \) ...................... \(AB\)

5. The median of an isosceles triangle from the vertex angle ......................

**Questions (2): Choose the correct answer:**

1. If \(\triangle ABC\) is an equilateral triangle, then \(m(\angle B) = \) .............. \(^\circ\)
   
   (a) 30\(^\circ\) \hspace{1cm} (b) 60\(^\circ\) \hspace{1cm} (c) 70\(^\circ\) \hspace{1cm} (d) 90\(^\circ\)

2. The length of the side opposite to angle of 30\(^\circ\) in the right-angled triangle = ..............
   
   Hypotenuse.
   
   (a) \(\frac{1}{2}\) \hspace{1cm} (b) \(\frac{1}{3}\) \hspace{1cm} (c) \(\frac{1}{4}\) \hspace{1cm} (d) 2
3) If the measure of the vertex angle of the isosceles triangle equals 80°, then the measure of one of the base angles = .......... °.
(a) 60 (b) 40 (c) 30 (d) 50

4) The number of axes of symmetry of the isosceles triangle = .................
(a) 1 (b) 2 (c) 3 (d) 0

5) \( \triangle ABC \), \( m (\angle A) = 50^\circ \), \( m (\angle B) = 60^\circ \), then the longest side is .................
(a) \( \overline{AB} \) (b) \( \overline{BC} \) (c) \( \overline{AC} \)

**Questions (3): In the opposite figure:**

\( \triangle ABC \) is right-angled triangle at \( B \),
\( m (\angle C) = 30^\circ \), \( AB = 5 \) cm.

Find the length of \( AC \)

\[ \therefore m (\angle B) = .................^\circ \cdot m (\angle C) = .................^\circ \]

\[ \therefore AB = \frac{1}{2} x ................. \]

\[ \therefore AC = ................. \) cm

**Questions (4):**

A) \( \triangle ABC \), \( m (\angle A) = 40^\circ \), \( m (\angle B)= 75^\circ \), \( m (\angle C) = 65^\circ \), arrange the lengths of the sides of this triangle descendingly the order is : ............................................................

B) In the opposite figure:
\( AD \parallel \overline{BC} \)
Complete:
1- \( m(\angle B) = .................^\circ \)
2- The longest side of \( \triangle ABC \) is .................

**Questions (5): In the opposite figure Put (\checkmark) in front of the correct answer, (\times) for the incorrect answer:**

\( AB = AC = CD = AD = 10 \) cm, \( m (\angle BAC) = 70^\circ \)

1) \( m (\angle B) = 55^\circ \) ( )
2) \( m (\angle D) = 70^\circ \) ( )
3) \( m (\angle DCB) = 120^\circ \) ( )
4) \( AB + AD = 20 \) cm ( )
5) \( AB + BC = BC + CD \) ( )